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ON THE EXISTENCE OF STRONG NASH EQUILIBRIA

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On the Existence of Strong Nash Equilibria

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Abstract

This paper investigates the existence of strong Nash equilibria (SNE) in continuous and convex games. We show that the concavity and an additional condition on payoff functions, together with the compactness of strategy space, permit the existence of strong Nash equilibria. These conditions are satisfied in many economic games and are quite simple to check. We also characterize the existence of SNE by providing a necessary and sufficient condition. Moreover, we suggest a procedure that can be used to efficiently compute strong Nash equilibrium. The result is illustrated with an application to an economy with multilateral environmental externalities and to the simple oligopoly static model.

Keywords: Noncooperative game, strong Nash equilibrium, weak Pareto-efficiency.

1 Introduction

The concept of Nash equilibrium introduced by Nash [1951] is probably the most important behavioral solution concept in game theory. It is based on the idea of stability against any unilateral deviations. Nevertheless, Nash equilibrium has several important shortcomings. The most severe one is that many games have multiple equilibria, and players may not be clear about which one to focus on.[1] This leads to a selection problem. Many refinements, which can be used to separate

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[1]For a detailed discussion on the shortcomings of Nash equilibrium, see Maskin [2009].

reasonable equilibria from unreasonable ones, have been proposed, such as the perfect equilibrium (Selten [1975]), the proper equilibrium (Myerson [1978]), the sequential equilibrium (Kreps and Wilson [1982]), and more recently the strong Berge equilibrium (Larbani and Nessah [2001]). All these equilibria are related to one another in varying degrees. But, these refinements characterizing rational behavior can still include multiple equilibria. In many games, there are opportunities for joint deviations that are mutually beneficial for a subset of players.

This led Aumann [1959] to propose the idea of strong Nash equilibrium (SNE) which ensures a more restrictive stability than the Nash equilibrium. A strong equilibrium is defined as a strategic profile for which no subset of players has a joint deviation that strictly benefits all of them, while all other players are expected to maintain their equilibrium strategies. Since the deviating coalition can be a single player or the whole set of players, this implies that a SNE is a Nash equilibrium and is also weakly Pareto efficient (in the sense that there is no other profile strictly preferred by all players) among the Nash equilibria. Thus, a SNE is not only immune to unilateral deviations, but also to deviations by coalitions. We can then consider it as a refinement of Nash equilibrium which is weakly Pareto efficient.

The SNE has been used to study different noncooperative games as coalition formation (Hart and Kurz [1983,1984], Bernheim *et al.* [1987], Chander and Tulkens [1997], Le Breton and Weber [2005]), congestion games (Hotzman and Law-Yone [1997], Voorneveld *et al.* [1999]), voting models (Keiding and Peleg [2001], Brams and Sanver [2006] and Moulin [1982]), network formation (Matsubayachi and Yamakawa [2006]), production externality games (Moulin and Shenker [1992], Moulin [1994]), and many other economic situations: Abreu and Sen [1991], Tian [1999,2000,2003], Suh [1996,1997,2001,2003], Shin and Suh [1996], Hirai *et al.* [2006], Konishi *et al.* [1997a,1997b,1997c], Ma [2002], Milchtaich [1996], Yoshihara [1999], Nishihara [1999], Perry and Reny [1994], Ray [2001], Savvateev [2003], Slikker [2001], Voorneveld and Grahn [2002], Yi [1999] and Young [1998]. These series of examples reveal the explanatory power of such equilibrium concept.

However, the existence of strong Nash equilibrium is a problem. These contributions are also paradoxical since there does not exist a general theorem which establishes clear existence conditions for the SNE. Ichiishi [1981] introduced the notion of social coalitional equilibrium and proved its existence under a set of assumptions of a society[2]. The concept of social coalitional equilibrium extends the notion of social equilibrium introduced by Debreu [1952], to prevent deviations by coalitions. It can also be specialized to the strong Nash equilibrium. Then, the sufficient conditions for the existence of social coalitional equilibria are also sufficient for the

[2]Given a finite set of agents N, a society is a list of specified data $(\{X^j\}_{j\in N}, \{S^C\}_{C\in\mathfrak{P}}, \{u_C^j\}_{j\in C\in\mathfrak{P}}, \mathfrak{F})$.

existence of strong Nash equilibria. However, an assumption (Assumption 4) in Ichiishi [1981] is very hard to verify. [3] There are also several other studies on the existence of strong Nash equilibria in various specific environments: Guesnerie and Oddou [1981], Greenberg and Weber [1986] in models with local public goods, Greenberg and Weber [1986,1993] in voting models, Demange and Henriet [1991], Demange [1994] in industrial organization and location models, and Wako [1994] in a market with indivisible goods, Ichiishi [1993] in games with non-transferable utility functions, Konishi et al. [1997a,1997c] in finite games, Konishi *et al.* [1997b] in games without spillovers, Hotzman and Law-Yone [1997] and Rozenfeld and Tennenholtz [2006] in congestion games. Yet, there is no general theorem on the existence of strong Nash equilibria.

In the present paper we fill this gap by proposing some sufficient conditions for the existence of SNE for general games. We show that the concavity and an additional condition on payoff functions, together with the compactness of strategy space, permit the existence of strong Nash equilibria. Unlike the existing results, these conditions are satisfied in many economic games and are often quite simple to check. We also characterize the existence of SNE by providing a necessary and sufficient condition. Moreover, we suggest a procedure that can be used to efficiently compute strong Nash equilibrium. Our equilibrium existence results neither imply nor are implied by the existing results in the literature such as those in Ichiishi [1981].

The remainder of the paper is organized as follows. Section 2 presents the definitions of strong Nash equilibrium and its properties. Section 3 establishes sufficient conditions for the existence of a strong Nash equilibrium and provides a method for its computation. Section 4 is dedicated to an application of the main new result to an economy with multilateral environmental externalities and the simple oligopoly static model. Section 5 concludes.

2 Strong Nash Equilibrium and Its Properties

In this section, we give the definition of strong Nash equilibrium, its interpretations and some of its properties.

Consider the following noncooperative game in normal form:

$$G = (X_i,\ u_i)_{i\in I} \tag{2.1}$$

where $I = \{1, ..., n\}$ is the finite set of players, X_i is the set of strategies of player i and $u_i : X \longrightarrow \mathbb{R}$ is the payoff function of player i. Denote by $X = \prod_{i\in I} X_i$ the set of strategy profiles of

[3] Assumption 4: For every $x \in X$ and for every $v \in \mathbb{R}^n$, if there exists a balanced collection $\mathfrak{B}$ such that for each $C \in \mathfrak{B}$ there exists $y_C \in S_C(x)$ for which $v_j \leq u^j_C(x, y_C)$ for every $j \in C$, then there exist $P \in \mathfrak{F}$ and $z_D \in S_D(x)$ for every $D \in P$ such that $v_j \leq u^j_D(x, z_D)$.

the game and $u = (u_1, u_2, ..., u_n)$ the profile of utility functions.

Let $\Im$ denote the set of all coalitions (*i.e.*, nonempty subsets of I). For each coalition $S \in \Im$, denote by $-S = \{i \in I$ such that $i \notin S\}$ the remaining of coalition S. If S is reduced to a singleton $\{i\}$, we denote then by $-i$ the set of $-S$. We also denote by $X_S = \prod_{i \in S} X_i$ the set of strategies of players in coalition S. If $\{K_j\}_{j \in \{1,...,s\} \subset \mathbb{N}}$ is a partition of I, then any strategy profile $x = (x_1, ..., x_n) \in X$ can be written as $x = (x_{K_1}, x_{K_2}, .., x_{K_s})$ with $x_{K_i} \in X_{K_i}$.

We say that a game $G = (X_i,\ u_i)_{i \in I}$ is compact, convex and continuous, respectively if, for all $i \in I$, X_i is compact, convex, and u_i is continuous on X, respectively.

We say that a strategy profile $x^* \in X$ is a *Nash equilibrium* of game (2.1) if,

$$u_i(y_i, x^*_{-i}) \leq u_i(x^*)\ \forall i \in I,\ \forall y_i \in X_i.$$

DEFINITION 2.1 *(Aumann [1959])* A strategy profile $\overline{x} \in X$ is said to be *strong Nash equilibrium* (SNE) of game (2.1), if $\forall S \in \Im$, there does not exist any $y_S \in X_S$ such that

$$u_i(y_S, \overline{x}_{-S}) > u_i(\overline{x}),\ \ \forall\ i \in S. \tag{2.2}$$

DEFINITION 2.2 A strategy profile $\overline{x} \in X$ of game (2.1) is said to be *weakly Pareto efficient* if there does not exist any $y \in X$ such that $u_i(y) > u_i(\overline{x})$ for all $i \in I$.

A strategy profile is a strong Nash equilibrium if no coalition (including the grand coalition, *i.e.*, all players collectively) can profitably deviate from the prescribed profile. This definition immediately implies that any strong equilibrium is both weakly Pareto efficient and a Nash equilibrium. Indeed, if a coalition S deviates from its strategy $\overline{x}_S$ in some strong Nash equilibrium $\overline{x}$, then it cannot improve the earning of all its players at the same time if the rest of the players maintain its strategy $\overline{x}_{-S}$ of $\overline{x}$. This equilibrium is stable with regard to the deviation of any coalition.

DEFINITION 2.3 *(The Weakly α-Core)* A strategy profile $\overline{x} \in X$ is in the weakly α-core of game G (2.1), if $\forall S \in \Im$ and $\forall x_S \in X_S$, there exists a $y_{-S} \in X_{-S}$ such that

$$u_i(x_S, y_{-S}) \leq u_i(\overline{x}) \text{ for at least some } i \in S.$$

A strategy profile $\overline{x}$ is in the weakly α-core if for any coalition S and any deviation x_S of $\overline{x}_S$, the coalition of the remaining players ($-S$) can find a strategy y_{-S} such as in the new strategy (x_S, y_{-S}), the payoffs of at least one player in coalition S cannot be better than those in the strategy $\overline{x}$ (for all the players of the coalition S at the same time).

DEFINITION 2.4 *(The Weakly β-Core)* A strategy profile $\overline{x} \in X$ is in the weakly β-core of game G (2.1), if $\forall S \in \Im$, there exists a $y_{-S} \in X_{-S}$ such that for every $x_S \in X_S$,

$$u_i(x_S, y_{-S}) \leq u_i(\overline{x}) \text{ for at least some } i \in S.$$

For any coalition S, the coalition of players $-S$ possesses a strategy y_{-S} which prevents all deviations of the coalition S of the strategy $\overline{x}$. Thus stability property of an outcome in the weak β-core is stronger than that of the weak α-core: a deviating coalition S can be countered by the complement coalition $-S$ even if the players of S keep secret their joint strategy X_S.

DEFINITION 2.5 *(The k-Equilibrium)* A strategy profile $\overline{x} \in X$ is said to be k-equilibrium ($k \in \{1, 2, ..., \#I\}$) of game G (2.1), if for all coalitions S with $\#S = k$, there does not exist any $y_S \in X_S$ such that

$$u_i(y_S, \overline{x}_{-S}) > u_i(\overline{x}), \quad \forall\, i \in S.$$

No k-players' coalition can make all these players win at the same time by deviating from the strategy $\overline{x}$.

We deduce the following properties:

1. SNE is a Nash equilibrium. It is sufficient to consider $S = \{i\}$ in Definition 2.1.
2. SNE is weakly Pareto optimal. It is sufficient to consider $S = I$ in Definition 2.1.
3. SNE is an element in the weakly α-core set. It is sufficient to consider $y_{-S} = \overline{x}_{-S}, \forall S \in \Im$ in Definition 2.3.
4. SNE is an element in the weakly β-core set. It is sufficient to consider $y_{-S} = \overline{x}_{-S}, \forall S \in \Im$ in Definition 2.4.
5. SNE is also a k-equilibrium, $\forall k \in \{1, 2, ..., n\}$. It is sufficient to consider $\forall S \in \Im$ with $\#S = k$ in Definition 2.1.

The following lemma characterizes the strong Nash equilibrium of the game (2.1).

LEMMA 2.1 *The strategy profile $\overline{x} \in X$ is a strong Nash equilibrium of the game $G = (X_i,\, u_i)_{i \in I}$ if and only if for each $S \in \Im$, the strategy $\overline{x}_S \in X_S$ is weakly Pareto efficient for the sub-game $\langle X_S, f_j(., \overline{x}_{-S})_{j \in S}\rangle$.*

PROOF. It is a straightforward consequence of Definition 2.1. ■

3 Existence Results

In this section, we establish some sufficient conditions for the existence of strong Nash equilibria (SNE). To do so, we use the following g-fixed point Theorem given by Nessah and Chu [2004]. Let us briefly recall this theorem.

Lemma 3.1 *(Nessah and Chu [2004]) Let X be a nonempty compact set in a metric space E, and Y a nonempty convex and compact set in a locally convex Hausdorff space F. Let $g : X \to Y$ be a continuous function and $C : X \to 2^Y$ an upper hemicontinuous correspondence with nonempty closed and convex values. Suppose that the following conditions are met:*

(a) $g(X)$ is convex in Y;

(b) for each $g(x) \in \partial g(X)$, $C(x) \cap Z_{g(X)}(g(x)) \neq \emptyset$ where $Z_{g(X)}(g(x)) = \left[\overline{\bigcup_{h>0} \frac{g(X)-g(x)}{h}} + \{g(x)\}\right] \cap Y$.

Then, there exists $\overline{x} \in X$ such that $g(\overline{x}) \in C(\overline{x})$.

Let

$$\Delta_S = \{\lambda_S = (\lambda_1, ..., \lambda_{\#S}) \in \mathbb{R}^{\#S} : \lambda_i \geq 0,\ \forall i = 1, ..., \#S \text{ and} \sum_{j \in S} \lambda_j = 1\}$$

be the unit simplex of $\mathbb{R}^{\#S}$ ($S \in \Im$), and let

$$\Delta = \{\lambda = (\lambda_S, S \in \Im) : \lambda_S \in \Delta_S\}, \quad \widehat{X} = \prod_{S \in \Im} X_S.$$

Define the correspondence $C_S : X_{-S} \times \Delta_S \to 2^{X_S}$ by

$$C_S(x_{-S}, \lambda_S) = \{z_S \in X_S : \sup_{y_S \in X_S} \sum_{i \in S} \lambda_{i,S} u_i(x_{-S}, y_S) \leq \sum_{i \in S} \lambda_{i,S} u_i(x_{-S}, z_S)\},$$

and then the correspondence $C : X \times \Delta \to 2^{\widehat{X}}$ by

$$x \mapsto C(x, \lambda) = \{\widehat{z} = (z_S, S \in \Im) \in \widehat{X} : z_S \in C_S(x_{-S}, \lambda_S)\}.$$

Define the function $\phi : X \to \widehat{X}$ by

$$\phi(x) = ((x_S) : \ S \in \Im).$$

We then have the following lemma.

Lemma 3.2 *Suppose that for all $i \in I$, X_i is convex and compact. Then we have:*

(a) *The function ϕ is continuous on X.*

(b) *The set $\phi(X)$ is convex and compact.*

PROOF. The continuity of function ϕ is a consequence of its definition and the construction of the set $\widehat{X}$. The compactness of the set $\phi(X)$ is a consequence of Weierstrass Theorem. The convexity of $\phi(X)$ is a consequence of the linearity of ϕ, which is easily verified. ■

By Lemma 3.1, we can establish the following existence theorem on strong Nash equilibria of the game $G = (X_i,\ u_i)_{i\in I}$.

THEOREM 3.1 *Let X_i, $\forall i \in I$, be a nonempty convex and compact subset of a locally convex Hausdorff space and u_i, $i \in I$, be continuous and concave on X. Suppose there exists $\lambda \in \Delta$ such that for each $x \in X$, there exist $z \in X$ and $a > 0$ such that*

$$az_S + (1-a)x_S \in C_S(x_{-S}, \lambda_S),\ \forall S \in \Im. \tag{3.1}$$

Then, game (2.1) *possesses at least one strong Nash equilibrium.*

PROOF. We prove step by step that the functions ϕ and C defined by $\phi(x) = ((x_S) : S \in \Im)$ and $C(x,\lambda) = \{\widehat{z} = (z_S : S \in \Im) \in \widehat{X} : z_S \in C_S(x_{-S}, \lambda_S)\}$, respectively, satisfy the conditions of Lemma 3.1:

1) $\forall x \in X$, $C(x,\lambda) \neq \emptyset$. Indeed, for any $x \in X$, the function $y_S \mapsto \sum_{j\in S} \lambda_{j,S} u_j(y_S, x_{-S})$, $S \in \Im$ is continuous on the compact X_S and by the Weierstrass Theorem, there exists $\overline{z}_S \in X_S$ such that

$$\max_{y_S\in X_S} \sum_{j\in S} \lambda_{j,S} u_j(y_S, x_{-S}) = \sum_{j\in S} \lambda_{j,S} u_j(\overline{z}_S, x_{-S}),\ \textit{i.e. } \overline{z}_S \in C_S(x_{-S}, \lambda_S).$$

Hence $\widehat{z} = (\overline{z}_S, S \in \Im) \in C(x,\lambda)$ and consequently $C(x,\lambda) \neq \emptyset$.

2) $\forall x \in X$, $C(x,\lambda)$ is convex in $\widehat{X}$. Indeed, let $x \in X$ and $\overline{z}$, $\overline{\overline{z}}$ be two elements of $C(x,\lambda)$ and $\theta \in [0,1]$. We want to prove that $\theta\overline{z} + (1-\theta)\overline{\overline{z}} \in C(x,\lambda)$.
Since $\overline{z}$ and $\overline{\overline{z}}$ are two elements in $C(x,\lambda)$, we then have: for each $S \in \Im$:

$$\max_{y_S\in X_S} \sum_{j\in S} \lambda_{j,S} u_j(y_S, x_{-S}) \leq \sum_{j\in S} \lambda_{j,S} u_j(\overline{z}_S, x_{-S})$$

and

$$\max_{y_S\in X_S} \sum_{j\in S} \lambda_{j,S} u_j(y_S, x_{-S}) \leq \sum_{j\in S} \lambda_{j,S} u_j(\overline{\overline{z}}_S, x_{-S}).$$

Hence,

$$\max_{y_S\in X_S}\sum_{j\in S}\lambda_{j,S}u_j(y_S,x_{-S})\le \min\left\{\sum_{j\in S}\lambda_{j,S}u_j(\overline{z}_S,x_{-S}),\ \sum_{j\in S}\lambda_{j,S}u_j(\overline{\overline{z}}_S,x_{-S})\right\}.$$

The concavity of function u_i and the last inequality imply that

$$\max_{y_S\in X_S}\sum_{j\in S}\lambda_{j,S}u_j(y_S,x_{-S})\le \sum_{j\in S}\lambda_{j,S}u_j(\theta\overline{z}_S+(1-\theta)\overline{\overline{z}}_S,x_{-S}),\ \forall\theta\in[0,1].$$

Thus, $\theta\overline{z}+(1-\theta)\overline{\overline{z}}\in C(x,\lambda)$.

3) C is upper hemicontinuous over X. Note that X is compact, and thus $\widehat{X}$ is compact (Tychonoff Theorem). Thus, to prove that C is upper hemicontinuous on X, it suffices to prove that $Graph(C)\subset X\times\widehat{X}$ is closed.
To see this, let $(x,\widehat{z})\in\overline{Graph(C)}$. Then there exists a sequence $\{(x^p,\widehat{z}^p)\}_{p\ge1}$ in $Graph(C)$ which converges to $(x,\widehat{z})$.
Hence, we have $\forall p\ge1$, $\widehat{z}^p\in C(x^p,\lambda)$, *i.e.*,

$$\max_{y_S\in X_S}\sum_{j\in S}\lambda_{j,S}u_j(y_S,x^p_{-S})\le\sum_{j\in S}\lambda_{j,S}u_j(z^p_S,x^p_{-S}),\ \forall S\in\Im.$$

Then, by the continuity of functions u_i, as $p\to\infty$, we have

$$\max_{y_S\in X_S}\sum_{j\in S}\lambda_{j,S}u_j(y_S,x_{-S})\le\sum_{j\in S}\lambda_{j,S}u_j(z_S,x_{-S}),\ \forall S\in\Im,$$

i.e., $\widehat{z}\in C(x,\lambda)$, hence $(x,\widehat{z})\in Graph(C)$ which means that $Graph(C)$ is closed in $X\times\widehat{X}$. Thus the function C is upper hemicontinuous on X.

4) For each $\phi(x)\in\partial\phi(X)$, $C(x,\lambda)\cap Z_{\phi(X)}(\phi(x)\neq\emptyset$ where $Z_{\phi(X)}(\phi(x))=\left[\overline{\bigcup_{h>0}\frac{\phi(X)-\phi(x)}{h}}+\{\phi(x)\}\right]\cap\widehat{X}=\left[\overline{\bigcup_{h>0}\{h[\phi(u)-\phi(x)],\ u\in X\}}+\{\phi(x)\}\right]\cap\widehat{X}$. Let $x\in X$ such that $\phi(x)\in\partial\phi(X)$. According to Condition (3.1) of Theorem 3.1, we have $\exists z\in X$ and $\exists a>0$ such that $az_S+(1-a)x_S\in C_S(x_{-S},\lambda_S)$, $\forall S\in\Im$. Thus, $\phi(az+(1-a)x)=a\phi(z)+(1-a)\phi(x)\in C(x,\lambda)$ (because ϕ is linear). Since $a[\phi(z)-\phi(x)]\in\frac{\phi(X)-\{\phi(x)\}}{1/a}\subset\overline{\bigcup_{h>0}\frac{\phi(X)-\{\phi(x)\}}{h}}$, then $a\phi(z)+(1-a)\phi(x)=a[\phi(z)-\phi(x)]+\phi(x)\in Z_{\phi(X)}(\phi(x))$. Therefore, $a\phi(z)+(1-a)\phi(x)\in C(x,\lambda)\cap Z_{\phi(X)}(\phi(x))$, *i.e.* $C(x,\lambda)\cap Z_{\phi(X)}(\phi(x))\neq\emptyset$.

From 1)-4), we conclude that the correspondence C satisfies all conditions of Lemma 3.1. Consequently, $\exists\overline{x}\in X$ such that $\phi(\overline{x})\in C(\overline{x},\lambda)$, *i.e.*, $\forall S\in\Im$, $\overline{x}_S\in C_S(\overline{x}_{-S},\lambda_S)$. Therefore, $\forall S\in\Im$, $\forall y_S\in X_S$, we have:

$$\sum_{j\in S}\lambda_{j,S}u_j(y_S,\overline{x}_{-S})\le\sum_{j\in S}\lambda_{j,S}u_j(\overline{x}_S,\overline{x}_{-S})=\sum_{j\in S}\lambda_{j,S}u_j(\overline{x}).\qquad(3.2)$$

Now we prove that $\forall S \in \Im$, $\overline{x}_S$ is weakly Pareto efficient to the sub-game $\langle X_S, u_j(., \overline{x}_{-S})_{j\in S}\rangle$.
Suppose that $\exists S_0 \in \Im$ such that $\overline{x}_{S_0}$ is not weakly Pareto efficient to the sub-game $\langle X_{S_0}, u_j(., \overline{x}_{-S_0})_{j\in S_0}\rangle$. Then, there exists $\widetilde{y}_{S_0} \in X_{S_0}$ such that:

$$\forall j \in S_0,\ u_j(\widetilde{y}_{S_0}, \overline{x}_{-S_0}) > u_j(\overline{x}). \tag{3.3}$$

System (3.3), together with $\lambda \in \Delta$ implies that $\sum\limits_{j\in S_0} \lambda_{j,S} u_j(\widetilde{y}_{S_0}, \overline{x}_{-S_0}) > \sum\limits_{j\in S_0} \lambda_{j,S} u_j(\overline{x})$. This contradicts inequality (3.2) for $S = S_0$ and $y_S = \widetilde{y}_{S_0}$ in (3.2). Hence $\overline{x}_S$ is weakly Pareto efficient to the sub-game $\langle X_S, u_j(., \overline{x}_{-S})_{j\in S}\rangle$, $\forall S \in \Im$, and consequently, by Lemma 2.1 it is a strong Nash equilibrium. The proof is completed. ■

EXAMPLE 3.1 Consider a game with $n = 2$, $I = \{1, 2\}$, $X_1 = [1/3, 2]$, $X_2 = [3/4, 2]$, and

$$u_1(x) = -x_1^2 + x_2 + 1,$$
$$u_2(x) = x_1 - x_2^2 + 1.$$

Since X is compact and convex and payoff functions are continuous and concave on X, we only need to show that the condition in (3.1) is also satisfied so that we know there exists a strong Nash equilibrium. To see this, let $x = (x_1, x_2)$ and $\Im = \{\{1\}, \{2\}, \{1, 2\}\}$. Then there exists $\lambda = (1, 1, (0.6, 0.4))$ such that:

1) for $S = \{1\}$ and $\lambda_S = 1$, we have $\max\limits_{y_1\in X_1} u_1(y_1, x_2) = \max\limits_{y_1\in X_1} (-y_1^2 + x_2 + 1) = -(1/3)^2 + x_2 + 1$.
2) for $S = \{2\}$ and $\lambda_S = 1$, we have $\max\limits_{y_2\in X_2} u_2(x_1, y_2) = \max\limits_{y_2\in X_2} (x_1 - y_2^2 + 1) = x_1 - (3/4)^2 + 1$.
3) for $S = \{1, 2\}$ and $\lambda_S = (0.6, 0.4)$, we have $\max\limits_{(y_1,y_2)\in X} [0.6u_1(y_1, y_2) + 0.4u_2(y_1, y_2)] = \max\limits_{(y_1,y_2)\in X} [-0.6y_1^2 + 0.4y_1 - 0.4y_2^2 + 0.6y_2 + 1] = [-0.6(1/3)^2 + 0.4(1/3) - 0.4(3/4)^2 + 0.6(3/4) + 1]$.

Thus, for all $x \in X$, there exist $z = (1/3, 3/4) \in X$ and $a = 1$ such that

$$az_S + (1 - a)x_S \in C_S(x_{-S}, \lambda_S),\ \forall S \in \Im.$$

Therefore, by Theorem 3.1, the game has a strong Nash equilibrium.

In the following, we characterize the existence of strong Nash equilibria by providing a necessary and sufficient condition. To do so, define a function $\digamma : X \times \Delta \times \widehat{X} \longrightarrow \mathbb{R}$ by

$$\digamma(x, \lambda, \widehat{y}) = \sum_{S\in\Im} \sum_{i\in S} \lambda_i \{u_i(x_{-S}, y_S) - u_i(x)\},$$

where $\widehat{X} = \prod_{S\in\Im} X_S$.

Note that, by the definition of F, we have

$$\forall x \in X,\ \forall \lambda \in \Delta,\ \max_{\widehat{y}\in\widehat{X}} F(x,\lambda,\widehat{y}) \geq 0. \tag{3.4}$$

Indeed, for $x \in X$ and $\lambda \in \Delta$, letting $\widehat{y} = \phi(x) = (x_S,\ S \in \Im)$,[4] we have $F(x,\lambda,\widehat{y}) = 0$, and consequently, $\max_{\widehat{y}\in\widehat{X}} F(x,\lambda,\widehat{y}) \geq 0$ for $\forall (x,\lambda) \in X \times \Delta$.

We then have the following theorem.

THEOREM 3.2 *Suppose that X_i, $\forall i \in I$, is a nonempty convex and compact subset of a locally convex Hausdorff space, the function u_i, $\forall i \in I$, is continuous and strictly concave on X. Let*

$$\alpha = \min_{\lambda\in\Delta} \min_{x\in X} \max_{\widehat{y}\in\widehat{X}} F(x,\lambda,\widehat{y}).$$

Then, the game (2.1) *has at least one strong Nash equilibrium if and only if $\alpha = 0$.*

PROOF. *Sufficiency.* According to Assumption of Theorem 3.2, for all $x \in X$ and $\lambda \in \Delta$ the maximum of the function $F(x,\lambda,.)$ over $\widehat{X}$ and $\min_{x\in X} \min_{\lambda\in\Delta} \max_{\widehat{y}\in\widehat{X}} F(x,\lambda,\widehat{y})$ exist.

Suppose that $\alpha = 0$. Since the functions $x \mapsto F(x,\lambda,\widehat{y})$ and $\lambda \mapsto F(x,\lambda,\widehat{y})$ are continuous over compact X and Δ, respectively, then Weierstrass Theorem implies there exist $\overline{x} \in X$ and $\overline{\lambda} \in \Delta$ such that $\alpha = \max_{\widehat{y}\in\widehat{X}} F(\overline{x},\overline{\lambda},\widehat{y}) = 0$, and this equality implies $\forall \widehat{y} \in \widehat{X}$, $F(\overline{x},\overline{\lambda},\widehat{y}) = \sum_{S\in\Im} \sum_{i\in S} \overline{\lambda}_{i,S}\{u_i(\overline{x}_{-S}, y_S) - u_i(\overline{x})\} \leq 0$.

For any arbitrarily fixed $S \in \Im$, we have $\forall \widehat{y} \in \widehat{X}$,

$$F(\overline{x},\overline{\lambda},\widehat{y}) = \sum_{i\in S} \overline{\lambda}_{i,S}\{u_i(\overline{x}_{-S}, y_S) - u_i(\overline{x})\} + \sum_{K\in\Im, K\neq S} \sum_{i\in K} \overline{\lambda}_{i,S}\{u_i(\overline{x}_{-K}, y_K) - u_i(\overline{x})\} \leq 0.$$

For $\widehat{y} \in \widehat{X}$ such that y_S is arbitrarily chosen in X_S and $y_K = \overline{x}_K, \forall K \neq S$, we have $\sum_{K\in\Im, K\neq S} \sum_{i\in K} \overline{\lambda}_{i,S}\{u_i(\overline{x}_{-K}, y_K) - u_i(\overline{x})\} = 0$. Then, from the last inequality, we deduce that $\forall y_S \in X_S,\ \sum_{i\in S} \overline{\lambda}_{i,S} u_i(\overline{x}_{-S}, y_S) \leq \sum_{i\in S} \overline{\lambda}_{i,S} u_i(\overline{x})$. Since S is arbitrarily chosen in $\Im$, then

$$\forall y_S \in X_S,\ \sum_{i\in S} \overline{\lambda}_{i,S} u_i(\overline{x}_{-S}, y_S) \leq \sum_{i\in S} \overline{\lambda}_{i,S} u_i(\overline{x}),\ \forall S \in \Im. \tag{3.5}$$

Now we prove that $\forall S \in \Im$, $\overline{x}_S$ is weakly Pareto efficient for the sub-game $\langle X_S, u_j(., \overline{x}_{-S})_{j\in S}\rangle$.

Suppose that $\exists S_0 \in \Im$ such that $\overline{x}_{S_0}$ is not weakly Pareto efficient for the sub-game $\langle X_{S_0}, u_j(., \overline{x}_{-S_0})_{j\in S_0}\rangle$. Then, there exists $\widetilde{y}_{S_0} \in X_{S_0}$ such that:

$$\forall j \in S_0,\ u_j(\widetilde{y}_{S_0}, \overline{x}_{-S_0}) > u_j(\overline{x}). \tag{3.6}$$

[4]The function ϕ is defined on page 6.

System (3.6) implies that $\sum_{j\in S_0}\overline{\lambda}_{j,S_0}u_j(\widetilde{y}_{S_0},\overline{x}_{-S_0}) > \sum_{j\in S_0}\overline{\lambda}_{j,S_0}u_j(\overline{x})$ ($\overline{\lambda}_{j,S_0}\geq 0$ and $\sum_{j\in S_0}\overline{\lambda}_{j,S_0} = 1$). This contradicts inequality (3.5) for $S = S_0$ and $y_S = \widetilde{y}_{S_0}$ in (3.5). Hence, $\overline{x}_S$ is weakly Pareto efficient for the sub-game $\langle X_S, u_j(.,\overline{x}_{-S})_{j\in S}\rangle$, $\forall S\in\Im$. Consequently, by Lemma 2.1, $\overline{x}_S$ is a strong Nash equilibrium. This proves the sufficiency.

Necessity: We will use the following result.

LEMMA 3.3 *(Moulin and Fogelman-Soulié [1979] pp 162) Suppose that X is convex in a vectorial space and the functions u_i, $i\in I$, are concave on X. Then, $\overline{x}\in X$ is a weakly Pareto efficient strategy profile of the game* (2.1) *if and only if there exists $\lambda\in\Delta_I$ such that* $\sup_{y\in X}\sum_{i\in I}\lambda_{i,I}u_i(y) = \sum_{i\in I}\lambda_{i,I}u_i(\overline{x})$.

Let $\overline{x}\in X$ be a strong Nash equilibrium of the game (2.1). According to Lemma 2.1, $\overline{x}_S$ is weakly Pareto efficient to the sub-game $\langle X_S, u_j(.,\overline{x}_{-S})_{j\in S}\rangle$, $\forall S\in\Im$. Since X_i, $i\in I$, is a nonempty convex and compact set and u_i, $i\in I$, is continuous and strictly concave on X, then according to Lemma 3.3, there exists $\overline{\lambda}_S\in\Delta_S$ such as $\max_{y_S\in X_S}\sum_{i\in S}\overline{\lambda}_{i,S}\{u_i(\overline{x}_{-S},y_S)-u_i(\overline{x})\} = 0$, $\forall S\in\Im$. This equality implies:

$$\max_{\widehat{y}\in\widehat{X}}\digamma(\overline{x},\overline{\lambda},\widehat{y}) = 0.$$

Thus, we have:

$$\alpha = \min_{x\in X}\min_{\lambda\in\Delta}\max_{\widehat{y}\in\widehat{X}}\digamma(x,\lambda,\widehat{y}) \leq \max_{\widehat{y}\in\widehat{X}}\digamma(\overline{x},\overline{\lambda},\widehat{y}) = 0. \quad (3.7)$$

Inequalities (3.4) and (3.7) imply $\alpha = 0$. This proves the necessity. ■

Theorem 3.2 actually provides a method of finding a SNE of game (2.1) under certain conditions (see Figure 1).

Suppose that all the conditions of Theorem 3.2 are satisfied.

Step 1. Calculate the value $\alpha = \min_{x\in X}\min_{\lambda\in\Delta}\max_{\widehat{y}\in\widehat{X}}\digamma(x,\lambda,\widehat{y})$.

Step 2.

If $\alpha > 0$, then the game $G = (X_i,\ u_i)_{i\in I}$ has no SNE.

If $\alpha = 0$, then the strategy profile $\overline{x}\in X$ such that $\min_{\lambda\in\Delta}\max_{\widehat{y}\in\widehat{X}}\digamma(\overline{x},\lambda,\widehat{y}) = 0$ is a SNE of the game $G = (X_i,\ u_i)_{i\in I}$.

Figure 1: Procedure for the determination of a SNE

By relaxing the strict concavity of function u_i, we obtain the following proposition.

PROPOSITION 3.1 *Let X_i, $\forall i \in I$, be a nonempty compact subset of a topological space, and the functions u_i, $\forall i \in I$, be continuous on X. Define the function $\alpha : \Delta \to \mathbb{R}$ by*

$$\alpha(\lambda) = \min_{x \in X} \max_{\widehat{y} \in \widehat{X}} \digamma(x, \lambda, \widehat{y}). \tag{3.8}$$

If there exists $\overline{\lambda} \in \Delta$ such that $\alpha(\overline{\lambda}) = 0$, then the game (2.1) *possesses at least one strong Nash equilibrium.*

PROOF. It is the same as that sufficiency in the proof of Theorem 3.2. ■

EXAMPLE 3.2 Let us again consider Example 3.1. In this case, $\widehat{X} = X_1 \times X_2 \times (X_1 \times X_2)$, $\widehat{y} = (a, b, (c, d)) \in X_1 \times X_2 \times (X_1 \times X_2)$ and $x = (u, v)$. Let $\lambda = (1, 1, (0.6, 0.4))$. Then, we have $\alpha_\lambda = \min_{x \in X} \max_{\widehat{y} \in \widehat{X}} \digamma(x, \lambda, \widehat{y}) = \min_{x \in X} \max_{\widehat{y} \in \widehat{X}} \{[u_1(a, v) - u_1(u, v)] + [u_2(u, b) - u_2(u, v)] + [0.6(u_1(c, d) - u_1(u, v)) + 0.4(u_2(c, d) - u_2(u, v))]\} = \min_{x \in X} \max_{\widehat{y} \in \widehat{X}} \{-a^2 - b^2 + (-0.6c^2 + 0.4c) + (-0.4d^2 + 0.6d) + (1.6u^2 - 0.4u) + (1.4v^2 - 0.6v)\}$.

Therefore, $\max_{a,c \in X_1, b,d \in X_2} \{-a^2 - b^2 + (-0.6c^2 + 0.4c) + (-0.4d^2 + 0.6d)\} = -55/144$ and $\min_{u \in X_1, v \in X_2} 1/2\{3u^2 - u + 3v^2 - v\} = 55/144$. The minimum is attained at $u = 1/3$ and $v = 3/4$. Thus, by Proposition 3.1, we know that $(1/3, 3/4)$ is a strong Nash equilibrium.

EXAMPLE 3.3 Consider the following game with $n = 2$, $I = \{1, 2\}$, $X_1 = X_2 = [-1, 1]$, and

$$u_1(x) = 3x_1 - x_2^2 + 4x_2,$$
$$u_2(x) = -x_1^2 + x_1 - 2x_2.$$

It is obvious to see that the functions u_i are strictly concave over the convex and compact X, $i = 1, 2$.

In this case, we have $\widehat{X} = X_1 \times X_2 \times (X_1 \times X_2)$. Letting $\widehat{y} = (a, b, (c, d)) \in X_1 \times X_2 \times (X_1 \times X_2)$ and $x = (u, v)$, we have $\alpha = \min_{(x,\lambda) \in X \times \Delta} \max_{\widehat{y} \in \widehat{X}} \digamma(x, \widehat{y}) = \min_{\lambda \in [0,1]} \min_{u,v \in [-1,1]} \max_{a,b,c,d \in [-1,1]} \{[u_1(a, v) - u_1(u, v)] + [u_2(u, b) - u_2(u, v)] + [\lambda(u_1(c, d) - u_1(u, v)) + (1 - \lambda)(u_2(c, d) - u_2(u, v))]\} = \min_{u,v \in [-1,1]} \min_{\lambda \in [0,1]} \max_{a,b,c,d \in [-1,1]} \{[3a - 2b] + [-(1 - \lambda)c^2 + (1 + 2\lambda)c] + [-\lambda d^2 + 2(3\lambda - 1)d] + [(1 - \lambda)u^2 - 2(2 + \lambda)u] + [\lambda v^2 + (4 - 6\lambda)v]\}$.

Consider the function $h : [0, 1] \to \mathbb{R}$ defined by $\lambda \mapsto h(\lambda) = \min_{u,v \in [-1,1]} \max_{a,b,c,d \in [-1,1]} \{[3a - 2b] + [-(1-\lambda)c^2 + (1+2\lambda)c] + [-\lambda d^2 + 2(3\lambda - 1)d] + [(1-\lambda)u^2 - 2(2+\lambda)u] + [\lambda v^2 + (4-6\lambda)v]\}$.

Recall that $\alpha = \min_{\lambda \in [0,1]} h(\lambda)$. $\min_{x \in X} \max_{\widehat{y} \in \widehat{X}} F(x, \widehat{y})$ is attained at

$$\begin{aligned}
\widetilde{a} &= 1, \\
\widetilde{u} &= 1, \\
\widetilde{b} &= -1, \\
\widetilde{c} &= \begin{cases} \frac{1+2\lambda}{2(1-\lambda)} & \text{if } 0 \le \lambda \le 1/4 \\ 1 & \text{if } 1/4 \le \lambda \le 1 \end{cases}, \\
\widetilde{d} &= \begin{cases} -1 & \text{if } 0 \le \lambda \le 1/4 \\ \frac{3\lambda-1}{\lambda} & \text{if } 1/4 \le \lambda \le 1/2 \\ 1 & \text{if } 1/2 \le \lambda \le 1 \end{cases}, \\
\widetilde{v} &= \begin{cases} -1 & \text{if } 0 \le \lambda \le 1/2 \\ \frac{3\lambda-1}{\lambda} & \text{if } 1/2 \le \lambda \le 1/2. \end{cases}
\end{aligned}$$

We then have:

$$h(\lambda) = \begin{cases} \frac{16\lambda^2-8\lambda+1}{4(1-\lambda)} & \text{if } 0 \le \lambda \le 1/4 \\ \frac{16\lambda^2-8\lambda+1}{\lambda} & \text{if } 1/4 \le \lambda \le 1/2 \\ \frac{-4\lambda^2+12\lambda-4}{\lambda} & \text{if } 1/2 \le \lambda \le 1. \end{cases}$$

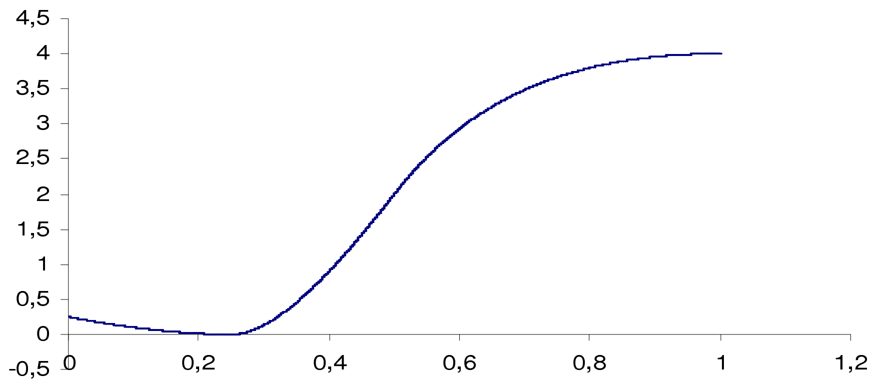

Figure 2: The graph of function h

Since $\alpha = \min_{\lambda \in [0,1]} h(\lambda) = h(1/4) = 0$ (see Figure 2.), by Theorem 3.2, the game has a strong Nash equilibrium which is $\overline{x} = (\widetilde{u}, \widetilde{v}) = (1, -1)$.

Remark 3.1 *The strict concavity of functions u_i, $i \in I$, of Theorem 3.2 is not necessary for the existence of SNE, because there are games not satisfying it and having a SNE.*

Example 3.4 Consider the following game with $n = 2$, $I = \{1, 2\}$, $X_1 = X_2 = [0, 1]$, and

$$u_1(x) = -x_1^2 - x_1 + x_2,$$
$$u_2(x) = 2x_1^2 + 3x_1 - x_2^2 - 3x_2.$$

It is obvious to see that the function u_2 is not strictly concave over the convex X.

In this case, we have $\widehat{X} = X_1 \times X_2 \times (X_1 \times X_2)$. Letting $\widehat{y} = (a, b, (c, d)) \in X_1 \times X_2 \times (X_1 \times X_2)$ and $x = (u, v)$, we have $\alpha = \min_{(x,\lambda)\in X\times\Delta} \max_{\widehat{y}\in\widehat{X}} F(x, \widehat{y}) = \min_{\lambda\in[0,1]} \min_{u,v\in[-1,1]} \max_{a,b,c,d\in[-1,1]} \{[u_1(a, v)-u_1(u, v)]+[u_2(u, b)-u_2(u, v)]+[\lambda(u_1(c, d)-u_1(u, v))+ (1-\lambda)(u_2(c, d)-u_2(u, v))]\} = \min_{u,v\in[-1,1]} \min_{\lambda\in[0,1]} \max_{a,b,c,d\in[-1,1]} \{[-a^2-a-b^2-3b]+[(2-3\lambda)c^2+ (3-4\lambda)c]+[-(1-\lambda)d^2+(-3+4\lambda)d]+[(-1+3\lambda)u^2-(2-4\lambda)u]+[(2-\lambda)v^2+(6-4\lambda)v]\}$.

Define $h : [0, 1] \to \mathbb{R}$ by

$$\begin{aligned} h(\lambda) &= \min_{u,v\in[-1,1]} \max_{a,b,c,d\in[-1,1]} \{[-a^2 - a - b^2 - 3b] + [(2 - 3\lambda)c^2 + (3 - 4\lambda)c] \\ &\quad +[-(1 - \lambda)d^2 + (-3 + 4\lambda)d] + [(-1 + 3\lambda)u^2 - (2 - 4\lambda)u] \\ &\quad +[(2 - \lambda)v^2 + (6 - 4\lambda)v]\}. \end{aligned}$$

Recall that $\alpha = \min_{\lambda\in[0,1]} h(\lambda)$. Then $\min_{x\in X} \max_{\widehat{y}\in\widehat{X}} F(x, \widehat{y})$ is attained at

$$\widetilde{a} = 0,$$
$$\widetilde{b} = 0,$$
$$\widetilde{v} = 0,$$
$$\widetilde{c} = \begin{cases} 1 & \text{if } 0 \le \lambda \le 7/10 \\ \frac{3-4\lambda}{2(3\lambda-2)} & \text{if } 7/10 \le \lambda \le 3/4 \\ 0 & \text{if } 3/4 \le \lambda \le 1 \end{cases},$$
$$\widetilde{d} = \begin{cases} 0 & \text{if } 0 \le \lambda \le 3/4 \\ \frac{4\lambda-3}{2(1-\lambda)} & \text{if } 3/4 \le \lambda \le 5/6 \\ 1 & \text{if } 5/6 \le \lambda \le 1 \end{cases},$$
$$\widetilde{u} = \begin{cases} 1 & \text{if } 0 \le \lambda \le 2/5 \\ \frac{1-2\lambda}{-1+3\lambda} & \text{if } 2/5 \le \lambda \le 1/2 \\ 0 & \text{if } 1/2 \le \lambda \le 1. \end{cases}$$

We then have:

$$h(\lambda) = \begin{cases} 2 & \text{if } 0 \le \lambda \le 2/5 \\ \frac{-25\lambda^2+26\lambda-6}{3\lambda-1} & \text{if } 2/5 \le \lambda \le 1/2 \\ 5 - 7\lambda & \text{if } 1/2 \le \lambda \le 7/10 \\ \frac{16\lambda^2-24\lambda+9}{4(3\lambda-2)} & \text{if } 7/10 \le \lambda \le 3/4 \\ \frac{16\lambda^2-24\lambda+9}{4(1-\lambda)} & \text{if } 7/10 \le \lambda \le 3/4 \\ 5\lambda - 4 & \text{if } 3/4 \le \lambda \le 1. \end{cases}$$

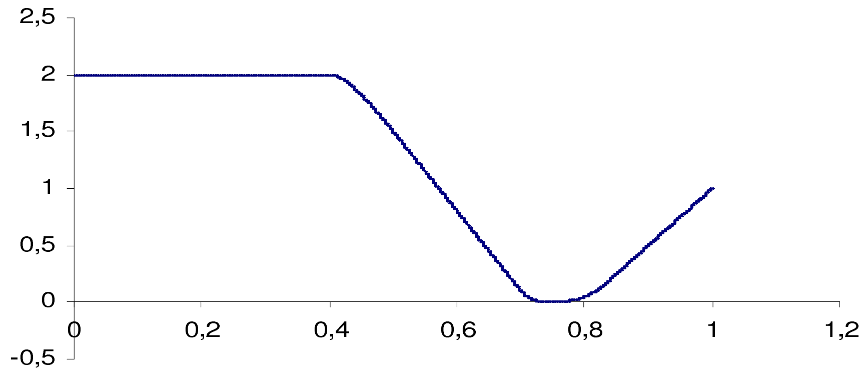

Figure 3: The graph of function h

Since $\alpha = \min_{\lambda \in [0,1]} h(\lambda) = h(3/4) = 0$ (see Figure 3.), by Proposition 3.1, the game has a strong Nash equilibrium as $\overline{x} = (\widetilde{u}, \widetilde{v}) = (0, 0)$.

4 Applications

4.1 Economy with Multilateral Environmental Externalities

Consider a simple economy with multilateral externalities and n agents, indexed by $i \in I = \{1, ..., n\}$. A consumption good $y_i \geq 0$ is produced from an input $e_i \geq 0$. The technology is described by a production function $y_i = g_i(e_i)$, and each agent's preference is presented by a quasilinear utility function $u_i(y_i, z) = y_i - v_i(z)$ where $v_i(z)$ is i's disutility function of the externality given by $z = \sum_{h \in I} e_h$. We make the following assumption.

Assumption 1. $g_i(e_i) - v_i(z)$ is strictly quasiconcave and differentiable over an interval $\prod_{i \in I} [0, e_i^0]$.

This model can be seen as an n-person noncooperative game where the i-th player chooses the input e_i. To do so, let

$$X_i = \{e_i \in \mathbb{R}_+ : \ 0 \leq e_i \leq e_i^0\}$$

be the strategy set of each player i, and

$$X_S = \{(e_i)_{i \in S} : \ 0 \leq e_i \leq e_i^0, \ i \in S\}$$

be the space of joint strategies of players in $S \in \Im$. Let X denote the space of joint strategies of all players, *i.e.*, $X = X_I$. For a strategy profile $[(e_1, ..., e_n)] \in X$, we choose $u_i(y_i, z) = y_i - v_i(z)$ with $z = \sum_{i \in I} e_i$ as the payoff for player i. Let $u = (u_1, ..., u_n)$. We have defined a noncooperative game:

$$G = \langle I, X, u \rangle. \tag{4.1}$$

We then have the following result.

PROPOSITION 4.1 *Suppose Assumption 1 is satisfied. Let*

$$\beta(\lambda) = \min_{e \in X} \max_{(d_S)_{S\in\Im} \in \prod\limits_{S\in\Im} X_S} \sum_{S\in\Im}\sum_{j\in S} \lambda_{j,S}\{[g_j(d_j) - g_j(e_j)] - [v_j(d_S + e_{-S}) - v_j(e)]\},$$

where $d_S + e_{-S} = \sum\limits_{j\in S} d_j + \sum\limits_{j\in -S} e_j$ *and* $e = \sum\limits_{j\in I} e_j$.
Then, the game (4.1) *possesses at least one strong Nash equilibrium if and only if there exists* $\lambda^* \in \Delta$ *such that* $\beta(\lambda^*) = 0$.

PROOF. It is the same as the proof of Theorem 3.2. ■

Consider $e \in X$, a coalition $S \in \Im$ and the subgame $G_S(e) = \langle S, X_S, u_j(., e_{-S})_{j\in S}\rangle$. According to Lemma 2.1, $\overline{e} \in X$ is a strong Nash equilibrium of the game (4.1) if and only if $\overline{e}_S \in X_S$ is weakly Pareto efficient for the subgame $G_S(\overline{e})$. By Lemma 3.3, $\overline{e}_S \in X_S$ is weakly Pareto efficient for the subgame $G_S(\overline{e})$ if and only if there exists $\lambda_S \in \Delta_S$ such that

$$\sup_{d_S\in X_S} \sum_{i\in S} \lambda_{i,S}[g_i(d_i) - v_i(d_S + \overline{e}_{-S})] = \sum_{i\in S} \lambda_{i,S}[g_i(\overline{e}_i) - v_i(\overline{e})]$$

where $d_S + \overline{e}_{-S} = \sum\limits_{j\in S} d_j + \sum\limits_{j\in -S} \overline{e}_j$ and $\overline{e} = \sum\limits_{j\in I} \overline{e}_j$.

To characterize weak Pareto efficiency for the subgame $G_S(e)$, we get the first order conditions

$$\lambda_{j,S} g_j'(d_j) = \sum_{h\in S} \lambda_{h,S} v_h'(\sum_{i\in S} d_i + \sum_{i\in -S} e_i),\ j \in S,\ \lambda_S \in \Delta_S. \tag{4.2}$$

Consider two coalitions $S_1, S_2 \in \Im$ and player j such that $j \in S_1 \cap S_2$. Then, (4.2) implies:

$$\begin{cases} (1)\ \lambda_{j,S_1} g_j'(d_j^1) = \sum\limits_{h\in S_1} \lambda_{h,S_1} v_h'(\sum\limits_{i\in S_1} d_i^1 + \sum\limits_{i\in -S_1} e_i),\ \lambda_{S_1} \in \Delta_{S_1}. \\ (2)\ \lambda_{j,S_2} g_j'(d_j^2) = \sum\limits_{h\in S_2} \lambda_{h,S_2} v_h'(\sum\limits_{i\in S_2} d_i^2 + \sum\limits_{i\in -S_2} e_i),\ \lambda_{S_2} \in \Delta_{S_2}. \end{cases} \tag{4.3}$$

For a solution of the equation (1)-(2) in system (4.3), d_j^1 is not necessarily equal to d_j^2. This means that for $e \in X$ to be a strong Nash equilibrium, it is necessary that $d_j^1 = d_j^2 = ... = d_j^k = e_j$, for each $j \in S_1 \cap S_2 \cap ... \cap S_k$. From this analysis we deduce the following assumption.

Assumption 2. There exist $\lambda \in \Delta$ and $e \in X$ such that

$$\lambda_{j,S} g_j'(e_j) = \sum_{h\in S} \lambda_{h,S} v_h'(\sum_{i\in I} e_i),\ \forall j \in S,\ \forall S \in \Im. \tag{4.4}$$

Then, Theorem 3.1 can be reformulated as follows.

COROLLARY 4.1 *Suppose Assumptions 1 and 2 are satisfied.* [5] *Then, the game* (4.1) *possesses at least one strong Nash equilibrium.*

[5] The solutions of the following system are within the set $\prod\limits_{i\in S} [0, e_i^0]$, $j \in S$ and $\lambda_S \in \Delta_S$:

$$\lambda_{j,S} g_j'(e_j) = \sum_{h\in S} \lambda_{h,S} v_h'(\sum_{i\in I} e_i).$$

EXAMPLE 4.1 Let us consider in the game (4.1) $I = \{1, 2, ..., n\}$, $x = (x_1, ..., x_n)$ $e = (e_1, ..., e_n)$ and

$$g_i(e_i) = a_i, u_i(y_i, z) = y_i - v_i(z)$$
$$\text{with } v_i(z) = z^2 - bz + c, a_i, b > 0 \text{ and } c \geq 0.$$

It's clear that the Assumption 1 is satisfied. We prove now that the Assumption 2 is also satisfied. Consider $\lambda \in \Delta$ and $\overline{e} \in X$ defined as follows:

$$\lambda_{i,S} = \frac{1}{\#S} \quad \forall S \in \Im \text{ and } z = \sum_{i \in I} \overline{e}_i = \frac{b}{2}.$$

Let $S \in \Im$, then we have

$$\begin{cases} g_j'(e_j) = 0, & \forall j \in S \\ v_j'(z) = 2z - b, & \forall j \in S. \end{cases}$$

Thus, (4.4) becomes $2z - b = 0, \;\; \forall j \in S$. Therefore, each $\overline{e} \in X$ such that $\sum\limits_{i \in I} \overline{e}_i = \frac{b}{2}$ is a strong Nash equilibrium.

DEFINITION 4.1 (Chander and Tulkens [1997]): For the game (4.1), the joint strategy $\overline{e}$ is a strong Nash equilibrium if for all $S \in \Im$

$$\overline{e}_S \;\; \text{maximizes} \; \sum_{i \in S} g_i(e_i) + v_i(z)$$

where $z + \sum\limits_{i \in S} e_i = \begin{cases} 0 \;\; \text{if} \;\; S = I \\ -\sum\limits_{i \in -S} \overline{e}_i \;\; \text{if} \;\; S \neq I. \end{cases}$

This definition (Definition 4.1) is a particular case of the Definition 2.1. Indeed, let $\overline{e}$ be a strong Nash equilibrium. Then, $\overline{e}_S$ is weakly Pareto efficient for the sub-game $\langle S, X_S, u_j(., \overline{e}_{-S})_{j \in S} \rangle$, for all $S \in \Im$. By Lemma 2.1, $\overline{e}$ is a strong Nash equilibrium in the sense of Definition 2.1.

REMARK 4.1 *Chander and Tulkens [1997] showed that, for the game* (4.1) *there does not exist a strong Nash equilibrium. The assumptions underlying these results are not sufficient for the non-existence of a strong Nash equilibrium (See the following Example).*

EXAMPLE 4.2 Assume that in game (4.1) $n = 2$ and $I = \{1, 2\}$, $x = (x_1, x_2)$ and $e = (e_1, e_2)$. Let

$$g_1(e_1) = \begin{cases} 80\sqrt{e_1}, & \text{if } 0 \leq e_1 < 1 \\ 80, & \text{if } 1 \leq e_1, \end{cases}, \; g_2(e_2) = \begin{cases} 86\sqrt{e_2}, & \text{if } 0 \leq e_2 < 1 \\ 86, & \text{if } 1 \leq e_2, \end{cases}$$

$$u_1(x_1, z) = x_1 - v_1(z) \text{ with } v_1(z) = z^2 - 3z,$$
$$u_2(x_2, z) = x_2 - v_2(z) \text{ with } v_1(z) = z^2 - 2z.$$

It is easy to show that all of Chander and Tulkens's conditions are satisfied. But, the strategy profile $(1, 1)$ is a strong Nash equilibrium for the game (4.1).

4.2 Simple Oligopoly Static Model

This subsection is dedicated to examining a simple oligopoly model. We first recall the Cournot model in which the firms are quantity choosers producing a homogeneous good.

Let p be the market price of a perfectly homogeneous good produced by the n firms of an industry, q_i be the sales of the i-th firm, $q = (q_1, ..., q_n)$, and let $Q = \sum_{i=1}^{n} q_i$ be the total sales in the market. The inverse demand function is $p = F(Q)$. The cost for the i-th firm is given by $C_i(q_i)$. The profit of the i-th firm is then given by $\psi_i(q) = q_i F(Q) - C_i(q_i)$.

We make the following assumptions.

Assumption 3. $F(Q)$ and $C_i(q_i)$ are continuous and nonnegative on $Q \in [0, +\infty)$ and $q_i \in [0, +\infty)$, respectively.

Assumption 4. There exists $\overline{q}_i > 0$, $i = 1, ..., n$ such that $\psi_i(q)$ is strictly concave over $\prod_{i\in I}[0, \overline{q}_i]$.

Let $X_i = [0, \overline{q}_i]$, $X = \prod_{i\in I}[0, \overline{q}_i]$, $X_S = \prod_{i\in S}[0, \overline{q}_i]$, for each $S \in \Im$ and we write $\psi = (\psi_1, ..., \psi_n)$, we have defined a noncooperative game

$$G = \langle I, X, \psi \rangle. \tag{4.5}$$

PROPOSITION 4.2 *Suppose that Assumptions 3 and 4 are satisfied. Let*

$$\gamma(\lambda) = \min_{q\in X} \max_{(r_S, S\in\Im)\in \prod_{S\in\Im} X_S} \sum_{S\in\Im}\sum_{j\in S}\lambda_{j,S}\{\psi_j(r_S, q_{-S}) - \psi_j(q)]\}.$$

Then, the game (4.5) *possesses at least one strong Nash equilibrium if and only if there exists* $\lambda^* \in \Delta$ *such that* $\gamma(\lambda^*) = 0$.

PROOF. It is the same as the proof of Theorem 3.2. ■

We make the following assumption.

Assumption 5. There exist $\lambda \in \Delta$ and $q \in X$ such that

$$\lambda_{j,S}C_j'(q_j) = \lambda_{j,S}F(\sum_{i\in I} q_i) + F'(\sum_{i\in I} q_i)\sum_{h\in S}\lambda_{h,S}q_h, \ \forall j \in S, \ \forall S \in \Im. \tag{4.6}$$

COROLLARY 4.2 *Suppose Assumptions 3,4 and 5 are satisfied.* [6] *Then, the game* (4.5) *possesses at least one strong Nash equilibrium.*

EXAMPLE 4.3 Consider a game with $I = \{1, 2\}$, $q = (q_1, q_2)$ and

$$F(Q) = aQ^2 - bQ + c, C_i(q_i) = \theta_i q_i^2 \text{ for } i = 1, 2$$
$$\text{with } b^2 - 4ac < 0 \text{ and } a, b, \theta_i > 0 \text{ for } i = 1, 2.$$

Suppose that

$$\psi_1(q_1, q_2) = aq_1^3 + (2aq_2 - b - \theta_1)q_1^2 + (aq_2^2 - bq_2 + c)q_1$$
$$\psi_2(q_1, q_2) = aq_2^3 + (2aq_1 - b - \theta_2)q_2^2 + (aq_1^2 - bq_1 + c)q_2$$

are strictly concave over $\prod_{i \in I}[0, \overline{q}_i]$.

If $(4ac - b^2)(\frac{1}{\theta_1} + \frac{1}{\theta_2}) = 4b$, then there exists $\overline{q} = (\frac{4ac-b^2}{8a\theta_1}, \frac{4ac-b^2}{8a\theta_2})$ such that Assumption 5 holds. Indeed, let $\lambda = (1, 1, (\frac{1}{2}, \frac{1}{2})) \in \Delta$, $\overline{q}_1 = \frac{4ac-b^2}{8a\theta_1}$ and $\overline{q}_2 = \frac{4ac-b^2}{8a\theta_2}$, then $\overline{q}_1 + \overline{q}_2 = \frac{b}{2a}$, *i.e.* $F^{'}(\overline{q}) = 0$.

The system (4.6) implies

$$\begin{cases} (1)\ 2\theta_1\overline{q}_1 = F(\overline{q}) + \overline{q}_1 F^{'}(\overline{q}) \\ (2)\ 2\theta_2\overline{q}_2 = F(\overline{q}) + \overline{q}_2 F^{'}(\overline{q}) \\ (3)\ 0 = \frac{1}{2}[-2\theta_1\overline{q}_1 + F(\overline{q}) + \overline{q}_1 F^{'}(\overline{q})] + \frac{1}{2}\overline{q}_2 F^{'}(\overline{q}) \\ (4)\ 0 = \frac{1}{2}[-2\theta_2\overline{q}_2 + F(\overline{q}) + \overline{q}_2 F^{'}(\overline{q})] + \frac{1}{2}\overline{q}_1 F^{'}(\overline{q}). \end{cases} \tag{4.7}$$

Since $F^{'}(\overline{q}) = 0$, then system (4.8) becomes:

$$\begin{cases} (1)\ 2\theta_1\overline{q}_1 = \frac{4ac-b^2}{4a} \\ (2)\ 2\theta_2\overline{q}_2 = \frac{4ac-b^2}{4a} \\ . \end{cases} \tag{4.8}$$

Thus, $\overline{q}_1 = \frac{4ac-b^2}{8a\theta_1}$ and $\overline{q}_2 = \frac{4ac-b^2}{8a\theta_2}$ such that $F^{'}(\overline{q}_1, \overline{q}_2) = 0$. This condition is equivalent to $(4ac - b^2)(\frac{1}{\theta_1} + \frac{1}{\theta_2}) = 4b$. Therefore, $\overline{q}_1 = \frac{4ac-b^2}{8a\theta_1}$ and $\overline{q}_2 = \frac{4ac-b^2}{8a\theta_2}$ is a strong Nash equilibrium.

[6] The solutions of the following system are within the set $\prod_{i \in S}[0, e_i^0]$, $j \in S$ and $\lambda_S \in \Delta_S$:

$$\lambda_{j,S} C_j^{'}(q_j) = \lambda_{j,S} F(\sum_{i \in I} q_i) + F^{'}(\sum_{i \in I} q_i) \sum_{h \in S} \lambda_{h,S} q_h.$$

5 Conclusion

In this paper, we studied the existence of strong Nash equilibria in general games. We proposed sufficient conditions (given by Theorem 3.1 and Proposition 3.1) for the existence of strong Nash equilibria. We also characterized the existence of strong Nash equilibria by providing a necessary and sufficient condition (Theorem 3.2). Our results would be useful for solving theoretical and practical problems from various domains.

References

Abreu, D., Sen, A.: Virtual implementation in Nash equilibrium. Econometrica **59**, 997-1021 (1991)

Aumann, J.P.: Acceptable points in general cooperative n-person games. In: Tucker, A.W., Luce, R.D. (eds.) Contributions to the Theory of Games IV. Princeton: Princeton University Press 1959

Berge, C.: Théorie génèrale des jeux à $n-$personnes. Gauthier Villars, Paris 1957

Bernheim, D.B., Peleg, B., Whinston, M.D.: Coalition-proof Nash equilibria: I. concepts. Journal of Economic Theory **42**, 1-12 (1987)

Brams, S.J., Sanver, M.R.: Critical strategies under approval voting: Who gets ruled in and ruled out. Electoral Studies **25**, 287-305 (2006)

Chander, P., Tulkens, H.: The core of an economy with multilateral environmental externalities. International Journal of Game Theory **26**, 379-401 (1997)

Demange, G.: Intermediate preferences and stable coalition structures. Journal of Mathematical Economics **23**, 45-58 (1994)

Demange, G., Henriet, D.: Sustainable oligopolies. Journal of Economic Theory **54**, 417-428 (1991)

Greenberg, J., Weber, S.: Multiparty equilibria under proportional representation. American Political Science Review **79**, 693-703 (1985)

Greenberg, J., Weber, S.: Strong Tiebout equilibrium under restricted preferences domain. Journal of Economic Theory **38**, 101-117 (1986)

Greenberg, J., Weber, S.: State coalition structures with unidimensional set of alternatives. Journal of Economic Theory **60**, 62-82 (1993)

Guesnerie, R., Oddou, C.: Second best taxation as a game. Journal of Economic Theory **60**, 67-91 (1981)

Hart, S., Kurz, M.: Stable coalition structures. In M.J. Holler (ed.), Coalitions and Collective Action, Vienna, Physica Verlag, 235-258 (1984)

Hart, S., Kurz, M.: Endogenous formation of coalitions. Econometrica **51**, 1047-1067 (1983)

Hirai, T., Masuzawaa, T., Nakayama, M.: Coalition-proof Nash equilibria and cores in a strategic pure exchange game of bads. Mathematical Social Sciences **51**, 162-170 (2006)

Hotzman, R., Law-Yone, N.: Strong equilibrium in congestion games. Games and Economic Bahavior **21**, 85-101 (1997)

Ichiishi, T.: The cooperative nature of the firm. London: Cambridge University Press 1993

Ichiishi, T.: A social coalitional equilibrium existence lemma. *Econometrica* **49**, 369-377 (1981)

Keiding, H., Peleg, B.: Stable voting procedures for committees in economic environments. Journal of Mathematical Economics **36**, 117-140 (2001)

Konishi H., Le Breton, M., Weber, S.: Equilibria in a model with partial rivalry. Journal of Economic Theory **72**, 225-237 (1997)

Konishi H., Le Breton, M., Weber, S.: Equivalence of strong and coalition-proof Nash equilibria in games without spillovers. Economic Theory **9**, 97-113 (1997)

Konishi H., Le Breton, M., Weber, S.: Equilibria in a model with partial rivalry. Journal of Economic Theory **72**, 225-237 (1997)

Kreps, D., Wilson, R.: Sequential equilibria. Econometrica **50**, 863-894 (1982)

Larbani, M., Nessah, R.: Sur l'équilibre fort selon Berge "Strong Berge equilibrium". RAIRO Operations Research **35**, 439-451 (2001)

Le Breton, M., Weber, S.: Stable partitions in a model with group-dependent feasible sets. Economic Theory **25**, 187-201 (2005)

Ma, J.: Stable matchings and the small core in Nash equilibrium in the college admissions problem. Review of Economic Design **7**, 117-134 (2002)

Maskin, E.: Nash equilibrium and mechanism design. Games and Economic Behavior, forthcoming.

Matsubayachi, N., Yamakawa, S.: A note on network formation with decay. Economics Letters **93**, 387-392 (2006)

Milchtaich, T.: Congestion models with player specific payoff functions. Games and Economic Behavior **13**, 111-124 (1996)

Moulin, H.: Serial cost sharing of excludable public goods. Review of Economic Studies **61**, 305-325 (1994)

Moulin, H., Shenker, S.: Serial cost sharing. Econometrica **60**, 1009-1037 (1992)

Moulin, H.: Voting with proportional veto power. Econometrica **50**, 145-162 (1982)

Moulin, H., Fogelman-Soulié, F.: La convexité dans les mathématiques de la décision. Hermann, Paris 1979

Myerson, R.B.: Refinements of the Nash equilibrium concept. International Journal of Game Theory **7**, 73-80 (1978)

Nash, J.F.: Noncooperative games. Annals of Mathematics **54**, 286-295 (1951)

Nessah, R., Chu, C.: Quasivariational equation. Mathematical Inequalities & Applications **7**, 149-160 (2004)

Nishihara, K.: Stability of the cooperative equilibrium in N-person prisoners' dilemma with sequential moves. Economic Theory **13**, 483-494 (1999)

Perry, M., Reny, P. J.: A noncooperative view of coalition formation and the core. Econometrica **62**, 795-817 (1994)

Ray, I.: On games with identical equilibrium payoffs. Economic Theory **17**, 223-231 (2001)

Rozenfeld, O., Tennenholtz, N.: Strong and correlated strong equilibria in monotone congestion games. In *Internet and Network Economics*, P. Spirakis et al. (Eds.): WINE 2006, Lecture Notes in Computer Science **4286**, 74-86, 2006.

Savvateev, A.: Strong equilibrium implementation for a principal with heterogeneous agents. EERC project No **00-103**, Economics Education and Research Consortium - Russia and Cis 17, 223-255

Selten, R.: Reexamination of the perfectness concept for equilibrium point in extensive games. International Journal of Game Theory **4**, 25-55 (1975)

Shin, S., Suh, S.C.: A mechanism implementing the stable rule in marriage problems. Economics Letters **51**, 185-189 (1996)

Slikker, M.: A one-stage model of link formation and payoff division. Games and Economic Behavior **34**, 153-175 (2001)

Suh, S.C.: Games implementing the stable rule of marriage problems in strong Nash equilibria. Social Choice and Welfare **20**, 33-39 (2003)

Suh, S.C.: An algorithm for verifying double implementability in Nash and strong Nash equilibria. Mathematical Social Sciences **41**, 103-110 (2001)

Suh, S.C.: Double implementation in Nash and strong Nash equilibria. Social Choice and Welfare **14**, 439-447 (1997)

Suh, S.C.: An algorithm for checking strong Nash implementability. Journal of Mathematical Economics **25**, 109-122 (1996)

Tian, G.: A solution to the problem of consumption externalities. Journal of Mathematical Economics **39**, 831-847 (2003)

Tian, G.: Implementation of balanced linear cost share equilibrium solution in Nash and strong Nash equilibria. Journal of Public Economics **76**, 239-261 (2000)

Tian, G.: Double implementation in economies with production technologies unknown to the designer. Economic Theory **13**, 689-707 (1999)

Wako, J.: Coalition-proofness the competitive allocations in a market with indivisible goods. Gakushuin University Discussion Paper, 1994

Voorneveld, M., Grahn, S.: Cost allocation in shortest path games. Mathematical Methods of Operations Research **56**, 323-340 (2002)

Voorneveld, M., Borm, P., Van-Megen, F., Tijs, S., Facchini, G.: Congestion games and potentials reconsidered. International Game Theory Review **1**, 283-299 (1999)

Yi, S.S.: On the coalition-proofness of the Pareto frontier of the set of Nash equilibria. Games and Economic Behavior **26**, 353-364 (1999)

Young, H.P.: Cost allocation, demand revelation, and core implementation. Mathematical Social Sciences **36**, 213-228 (1998)

Yoshihara, N.: Natural and double implementation of public ownership solutions in differentiable production economies. Review of Economic Design **4**, 127-151 (1999)