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INFEASIBILITIES AND DIRECTIONAL DISTANCE FUNCTIONS

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Infeasibilities and Directional Distance Functions

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Abstract

The purpose of this contribution is to highlight an underexplored property of the directional distance function, a recently introduced generalization of the Shephardian distance function. It diagnoses in detail the economic conditions under which infeasibilities may occur for the case of directional distance functions and explores whether there exist any solutions that remedy the problem in an economically meaningful way. This discussion is linked to determinateness as a property in index theory.

Keywords: directional distance function, shortage function, well-definedness, infeasibility, determinateness.
JEL: C43, D21, D24.

1 Introduction

The purpose of this contribution is to explore an underdeveloped property of a recent generalization of Shephard (1970) distance function, known as the directional distance function. Distance functions are employed in consumption and production theory. Luenberger (1992) defined the benefit function as a directional representation of preferences, which generalizes Shephard's (1970) input distance function defined in terms of the utility function. Luenberger (1995) introduced the shortage function as a transposition of the benefit function in a production context. Chambers, Chung and Färe (1996) relabel this same function as a directional distance function and since then it is commonly known by that name. The directional distance function generalizes existing distance functions by accounting for both input contractions and

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output improvements and it is dual to the profit function. Furthermore, the directional distance function is flexible due to the variety of direction vectors it allows for (see, e.g., Chambers, Färe and Grosskopf (1996)). Chambers, Chung and Färe (1996) analyze both the benefit function and the directional distance function in some depth and extend the composition rules of McFadden (1978) to these new concepts.

It is well known that in certain cases the directional distance function is not well-defined and achieves a value of infinity (see, e.g., Chambers, Chung and Färe (1996: pp. 409-410) or Luenberger (1995)). This is related to the property of determinateness in index theory, which can be loosely stated as requiring that an index remains well-defined (i.e., cannot become indeterminate or infinite) when any of its arguments become zero or infinity. Being one of Fisher's (1922) original axioms, determinateness has aroused some discussion. Swamy (1965) found it suspect and an eventual candidate to drop to guarantee consistency of the original Fisher (1922) tests, a view seemingly also shared by Eichhorn (1976). Samuelson and Swamy (1974) simply rejected determinateness. By contrast, Färe and Lyon (1981) specify conditions on technology that guarantee determinateness for an input price index. Thus, there are at least two fundamental attitudes with respect to determinateness in the index literature. First, reject determinateness and simply report any indeterminacies of indices found in practice. Second, accept determinateness and look for some conditions guaranteeing it.

This determinateness problem also crops up in the more recent literature on discrete-time productivity indices. Discrete-time Malmquist input- and output-oriented productivity indexes based upon Shephardian distance functions as general technology representations (Caves, Christensen and Diewert (1982)) have been made empirically tractable by Färe et al. (1995). But, some of the distance functions constituting this Malmquist index may well be infeasible when estimated upon general technologies using nonparametric estimators.[1] It is possible to show that almost all other recent discrete-time primal productivity indices and indicators may suffer from the same problem in a number of economic contexts.[2] Färe et al. (1995) avoid this problem by imposing a technology with a restrictive returns to scale assumption. However, Chambers and Pope (1996: 1364) strongly argue against restrictive returns to scale assumptions (e.g., constant returns to scale) that are only relevant for, e.g., a representative firm supposedly to be in long-run equilibrium. As above indicated, this could imply simply reporting the infeasibilities

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[1] Similar problems occur in static applications of the directional distance function when an observation is evaluated to a technology to which it need not belong. One example is the measurement of gains of diversification or specialisation when considering potential candidates for mergers (see Färe, Grosskopf and Lovell (1994)).

[2] Meanwhile more general primal productivity indicators have been proposed: e.g., Chambers and Pope (1996) define a Luenberger productivity indicator in terms of differences between directional distance functions. Notice that ”indicators” (”indexes”) denote productivity measures based on differences (ratios) (see Diewert (2005)).

when computing productivity indices and indicators.[3]

While it is true that the vast majority of empirical productivity studies employ deterministic, nonparametric technologies (see Varian (1984) and Banker and Maindiratta (1988)), our analysis is also valid for parametric specifications of technology.[4] Thus, the paper is phrased in terms of general technologies and does not privilege a specific estimation method. However, since the most popular estimation method employs nonparametric technologies we mostly use the word infeasibility as a manifestation of a lack of well-definedness throughout the paper.

The purpose of this contribution is to extend the Luenberger (1995) and Chambers, Chung and Färe (1996) analysis regarding the directional distance function by diagnosing the economic conditions under which infeasibilities may occur and by exploring whether there exist any solutions that could remedy the problem in an economically meaningful way.[5] Concurring with Chambers and Pope (1996), we do not follow Färe and Lyon (1981) by looking for eventual restrictions on technology. Instead, the analysis focuses on the choice of direction vector when using the directional distance function. This issue has hitherto been unexplored in the literature, probably since it arose with the definition of the directional distance function itself.

To develop these arguments, this contribution is structured as follows. Section 2 develops the basic definitions of the technology and the various distance functions. The next section states the general nature of the infeasibility problem in the definition of the directional distance function depending upon the choice of direction vector. A final section concludes. For convenience, the analysis is phrased in terms of production theory. The transposition of these results to the benefit function in consumption theory is immediate.

2 Technology and Distance Functions: Definitions

We first introduce the assumptions on technology and the definitions of the distance functions providing the components for computing productivity indicators. Production technology transforms inputs $x = (x_1, \cdots, x_n) \in \mathbb{R}^n_+$ into outputs $y = (y_1, \cdots, y_p) \in \mathbb{R}^p_+$. For each time period t, the production possibility set T summarises the set of all feasible input and output vectors and is defined as follows:

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[3] Unfortunately, few empirical studies explicitly report the prevalence of infeasibilities in, e.g., the Malmquist productivity index (Mukherjee, Ray and Miller (2001) is among the exceptions). Probably many researchers continue to assume that determinateness is crucial for index numbers.

[4] An example of an empirical productivity study using both nonparametric and parametric technologies is Atkinson, Cornwell and Honerkamp (2003).

[5] This analysis also applies to other general distance functions (e.g., McFadden's (1978) gauge function or the generalized distance function of Chavas and Cox (1999)).

$$T = \left\{(x, y) \in \mathbb{R}_+^{n+p};\ x \text{ can produce } y\right\}. \tag{2.1}$$

Throughout the paper technology satisfies the following conventional assumptions: (T.1) $(0,0) \in T,\ (0,y) \in T \Rightarrow\ y = 0$ i.e., no free lunch; (T.2) the set $A(x) = \{(u,y) \in T; \quad u \leqslant x\}$ of dominating observations is bounded $\forall x \in \mathbb{R}_+^n$, i.e., infinite outputs are not allowed with a finite input vector; (T.3) T is closed; and (T.4) $\forall (x,y) \in T,\ (x,-y) \leqslant (u,-v) \Rightarrow (u,v) \in T$, i.e., fewer outputs can always be produced with more inputs, and inversely (strong disposal of inputs and outputs). On one occasion, stronger assumptions (specifically, convexity) are needed.

While these assumptions are standard, it is possible to weaken some of these maintained axioms. For instance, strong input and output disposal may be (partially) replaced by the assumption of weak disposability. Notice that in such a case the resulting technologies may lead to even more from infeasibilities of the distance functions (see below), since the production possibility set is smaller. For instance, Jaenicke (2000) notices the issue of infeasibilities for technologies with weak disposal in the output dimensions.

Technology can be characterised by distance functions. To simplify the notation, we denote:

$$z = (x, y) \in T. \tag{2.2}$$

and

$$g = (h\,,\,k) \in (-\mathbb{R}_+^n) \times \mathbb{R}_+^p \tag{2.3}$$

which is partitioned in an input and an output direction vector h respectively k. The directional distance function involving a simultaneous input and output variation in the direction of a pre-assigned vector g is defined as:

Definition 2.1 *The function* $D_T : \mathbb{R}_+^{n+p} \times (-\mathbb{R}_+^n) \times \mathbb{R}_+^p \longrightarrow \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ *defined by*

$$D_T(z; g) = \begin{cases} \sup\{\delta \in \mathbb{R} : z + \delta g \in T\} & \text{if } z + \delta g \in T \text{ for some } \delta \in \mathbb{R} \\ -\infty & \text{otherwise} \end{cases}$$

is called the directional distance function in the direction of $g = (h\,,\,k)$.

Notice that distance functions are related to efficiency measures in that they measure deviations from the boundary of technology.

For the purpose of studying the problem of ill-defined productivity indicators, we distinguish between the standard case where the distance is achieved and the case where there is no way to achieve the distance. This distinction is fairly standard when defining distance functions (see, e.g., Chambers (2002)). Note that when no direction is selected and a point is part of the technology ($z \in T$), then $D_T(z; g) = +\infty$. This directional distance function (Chambers, Färe and Grosskopf (1996)) is a special case of the shortage function (Luenberger (1992)).

Note that the directional distance function is defined using a general directional vector g. However, sometimes we consider the special case: $h = -x$ and $k = y$, also known as the Farrell proportional distance function (Briec (1997)).[6] In the literature other direction vectors have been proposed (for instance, the translation function of Blackorby and Donaldson (1980) with $h = -1^n$ and $k = 0$, where 1^n is the n-dimensional unit vector). See Chambers, Färe and Grosskopf (1996) for additional choices of direction vectors.

The directional distance function generalises the Shephardian distance functions. For instance, the Shephardian input distance function results by setting $g = (h\,,\,0) = (-x, 0)$ and calculating $D_i(z) = [1 - D_T(z; -x, 0)]^{-1}$. The profit function $\Pi : \mathbb{R}_+^{n+p} \longrightarrow \mathbb{R} \cup \{\infty\}$ is defined as:

$$\Pi(w,p) = \sup_{(x,y)} \{p.y - w.x, (x,y) \in T\} \tag{2.4}$$

A dual formulation of the directional distance function is defined as follows:

Definition 2.2 *The function* $\bar{D}_T : \mathbb{R}_+^{n+p} \times (-\mathbb{R}_+^n) \times \mathbb{R}_+^p \longrightarrow \mathbb{R} \cup \{-\infty\}$ *defined by*

$$\bar{D}_T(z; g) = \inf_{(w,p) \geq 0} \{\Pi(w,p) - p.y + w.x : p.k - w.h = 1\}$$

is called the hyper-directional distance function in the direction of $g = (h\,,\,k)$.

Chambers, Chung and Färe (1998) prove duality between directional distance function and profit function when the former function is real-valued. In the latter case, $D_T(z; g) = \bar{D}_T(z; g)$. Clearly, this dual version of the directional distance function can be interpreted as a shadow profit function.

3 Directional Distance Function: Infeasibility and its Remedy

This section analyses the precise conditions under which infeasibilities may or may not occur. This is done for general points that need not be part of technology.

3.1 Infeasible Directions

We first define the concept of an infeasible direction for the directional distance function and focus on its relationship to a general production technology.

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[6] Axiomatic properties of this function are studied in Briec (1997) and Chambers, Chung and Färe (1998).

Definition 3.1 *Let* $g \in (-\mathbb{R}^n_+) \times \mathbb{R}^p_+$ *and for all* $z \in \mathbb{R}^{n+p}_+$ *let us denote:*

$$\Delta(z, g) = \{z + \delta g : \delta \in \mathbb{R}\}$$

the affine line generated from z *in the direction of* g*. We say that a direction* g *is:*
a) Infeasible at z *if:* $\Delta(z, g) \cap T = \emptyset$*;*
b) Interior if $g \in (-\mathbb{R}^n_{++}) \times \mathbb{R}^p_{++}$*.*

We can now state the following completely general result proving that for all technologies and for an arbitrary direction vector g there exists some point z such that the direction g is infeasible at point z. The proof below is based on the characteristic of the output set $P(x)$ that is bounded for all $x \in \mathbb{R}^p_+$. In particular, focusing on the at least two-dimensional output case, we show that for any non-zero direction there exists an input output vector such that the direction g is infeasible.

Proposition 3.2 *For all technologies* T *satisfying T1-T4 and* $g \in (-\mathbb{R}^n_+) \times \mathbb{R}^p_+$*, if the two following conditions hold:*
i) the number of output dimensions is greater than or equal to 2 ($p \geq 2$)*,*
ii) the output direction vector is non-zero ($k \neq 0$)*,*
then there exists some $z \in \mathbb{R}^{n+p}_+$ *such that the direction* g *is infeasible at* z*.*

Proof: We first consider the case where there is some $j \in \{1, \cdots, p\}$ such that $k_j = 0$. Since $p \geq 2$, this does not contradict $k_j \neq 0$. Now, consider some $x \in \mathbb{R}^n_+$ and let $P(x) = \{y \in \mathbb{R}^p_+ : x \text{ can produce } y\}$. Since $P(x)$ is compact, there exists some $\bar{y}$ such that $P(x) \subset \{v \in \mathbb{R}^p_+ : y \leq \bar{y}\}$. Let $y \in \mathbb{R}^p_+$ such that $y_j > \bar{y}_j$. Then, for all $\delta \in \mathbb{R}$, $y_j + \delta k_j = y_j > \bar{y}_j$. Thus, $y + \delta k \notin \{v \in \mathbb{R}^p_+ : y \leq \bar{y}\}$. Thus, $y + \delta k \notin P(x)$. Consequently, $(x, y) + \delta g \notin T$. Since $z = (x, y) \in \mathbb{R}^{n+p}_+$, we deduce that g is infeasible at z. Assume now that for all $j \in \{1, \cdots, p\}$, $k_j > 0$. Since $P(x)$ is compact, there is $j \in \{1, \cdots, p\}$ and some $y \in \mathbb{R}^p_+$ such that $y \in \{v \in \mathbb{R}^p : v_j = 0\}$ and $y \notin P(x)$. For all $\delta \geq 0$, $y + \delta k \in P(x) \Longrightarrow y \in P(x)$ (from the strong disposal assumption). This is a contradiction, thus for all $\delta \geq 0$, we have $y + \delta k \notin P(x)$. Moreover, since $y_j = 0$, $\delta < 0 \Longrightarrow y_j + \delta k \notin \mathbb{R}^p_+ \Longrightarrow y + \delta k \notin P(x)$. Thus, we deduce that $(x, y) + \delta g \notin T$, for all $\delta \in \mathbb{R}$. This ends the proof. $\square$

To illustrate this proposition, a numerical example is provided below for a simple three dimensional production technology with two outputs.

Example 3.3 *Assume that* $n = 1$ *and* $p = 2$*, and let us consider the production technology:*

$$T = \left\{(x, y_1, y_2) \in \mathbb{R}^3_+ : y_1 + y_2 \leq x\right\}$$

It is easy to check that T satisfies $T1 - T5$. Let $z = (1, 0, 2)$, clearly $z \notin T$. Moreover, let us consider the direction $g = (-1, 1, 1)$. The direction g is feasible at z if and only if the following system of linear inequalities has some solution:

$$\begin{cases} 1 - \delta & \geq & 0 \\ 0 + \delta & \geq & 0 \\ 2 + \delta & \geq & 0 \\ 2 + 2\delta & \leq & 1 - \delta \end{cases} \tag{3.1}$$

Clearly, the system (3.1) has no solution and thereby $D_T(z; g) = -\infty$.

Following Proposition 3.2, for a given technology with a number of outputs $p \geq 2$ and a given direction vector with non-null output direction, there always exists an input output vector such that the directional distance function takes the value $-\infty$.

Corollary 3.4 *For all production technologies T satisfying T1-T4 where $p \geq 2$ and all $g \in \mathbb{R}_+^{n+p}$, there exists $z \in \mathbb{R}_+^{n+p}$ such that $D(z; g) = -\infty$.*

This implies that one can always find a direction vector (with non-null output direction) which is infeasible for a given point z.

Corollary 3.5 *For all production technologies T satisfying T1-T4 where $p \geq 2$, there exists $g \in \mathbb{R}_+^{n+p}$ and $z \in \mathbb{R}_+^{n+p}$ such that $D(z; g) = -\infty$.*

Thus, this perfectly general result demonstrates that even the Luenberger productivity indicator, that employs the most general of distance functions, cannot avoid infeasibilities.

Furthermore, these results can serve to illustrate that some claims in the literature regarding the origin of the infeasibility problem are simply wrong. For instance, the output-oriented Malmquist productivity index can well be infeasible irrespective of the maintained returns to scale assumption on technology.[7] As another example, Jaenicke (2000: 257-258) suggests that imposing strong instead of weak output disposal on technology is sufficient to guarantee feasibility for a distance function with non-null output direction vector when constructing an output-oriented Malmquist index. This claim is erroneous, since even with the stronger assumption of strong output disposal maintained in this contribution it is impossible to rule out infeasibilities.

However, the above results are no longer valid when the output set in one-dimensional and the direction vector is semi-positive in inputs and positive in the single output, as it is stated in the next result.

Lemma 3.6 *Let T be a production technology satisfying T1-T4. If the output set is one-dimensional ($p = 1$) and if $g \in (-\mathbb{R}_+^n) \times \mathbb{R}_{++}$, then for all $z \in \mathbb{R}_+^{n+1}$, the direction g is feasible at z.*

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[7] The same remark would apply to the Luenberger output-oriented productivity indicator.

Proof: Assume that $z \notin T$. Let $\bar{\delta} = \frac{-y}{k}$. We have $z + \bar{\delta} g = (x + \bar{\delta} h, 0)$. Since $h \in -\mathbb{R}^n_+$, we deduce that $z + \bar{\delta} g \in \mathbb{R}^n_+ \times \{0\}$. Since $(0,0) \in T$, we deduce from the strong disposal assumption that $z + \bar{\delta} g \in T$. □

3.2 Infeasible Directions When the Output Direction Vector is Null

Now, we focus on the case where the output direction is null. Here, the eventual infeasibilities depend on the precise choice of the input direction.

We can formulate a first general result as follows:

Proposition 3.7 *Let* T *be a production technology satisfying* $T1 - T4$. *Let* $y \in \mathbb{R}^p_+$ *and assume that* $L(y) \neq \emptyset$. *Assume there exist* $i_0 \in \{1, \cdots, n\}$ *and* $\alpha_{i_0} \geq 0$ *such that for all* $u \in L(y)$, $u_{i_0} > \alpha_{i_0}$. *If* $g = (h, 0)$ *is a direction such that* $h_{i_0} = 0$, *then there exists some* $x \in \mathbb{R}^n_+$ *such that the direction* g *is infeasible at point* $z = (x, y)$ *for all* $y \in \mathbb{R}^p_+$.

Proof: We just consider the vector $x \in \mathbb{R}^n_+$ defined by

$$x_{i_0} = \begin{cases} 1 \text{ if } i \neq i_0 \\ \frac{\alpha_{i_0}}{2} \text{ if } i = i_0 \end{cases}$$

for $i = 1 \cdots n$. Now, let $h \in -\mathbb{R}^n_+$ such that $h_{i_0} = 0$. Now, it is clear that for all $\delta \in \mathbb{R}$, $x_{i_0} + \delta h_{i_0} = \frac{\alpha_{i_0}}{2} \leq \alpha_{i_0}$. But, since for all $u \in L(y)$, $u_{i_0} > \alpha_{i_0}$, we deduce that $x + \delta h \notin L(y)$. Consequently, for all vector $k \in \mathbb{R}^p_+$, and all $y \in \mathbb{R}^p_+$, $(x, y) + \delta g \notin T$, this ends the proof. □

Thus, whenever output direction is null, at least one input dimension is essential (i.e., there is a minimal level needed of this input to produce some outputs), and the input direction vector is not of full dimension in the essential input(s), there is always a point such that it is infeasible for a general technology.

A simple numerical example based on a Leontief technology is provided below showing that this type of infeasibility may well appear in a traditional parametric technology.

Example 3.8 *Assume that* $T = \{(x_1, x_2, y) \in \mathbb{R}^3_+ : y \leq \min\{x_1, x_2\}\}$. *If* $g = (-1, 0, 0)$, *then the direction* g *is infeasible at point* $(1, \frac{1}{2}, 1)$.

The next example focus on the more general Cobb-Douglas technology.

Example 3.9 *Assume that* $T = \{(x_1, x_2, y) \in \mathbb{R}^3_+ : y \leq x_1^{\theta_1} x_2^{\theta_2}\}$, *where* $\theta_1, \theta_2 > 0$. *If* $g = (-1, 0, 0)$, *then the direction* g *is infeasible at point* $(0, 1, 1)$.

In conclusion of both examples, it is clear that traditional parametric technology specifications are not immune to the infeasibility problem.

Next, we show that when the output correspondence is bounded[8], then for all input-oriented directions there exists an infeasible direction at some point in $\mathbb{R}_+^{n+p}$. Furthermore, if an output vector is attainable from an input vector and the direction vector is interior in the inputs, then the directional distance function is feasible.

Proposition 3.10 *Let T be a production technology satisfying $T1-T4$. We have the following properties:*
a) If P is a bounded correspondence, then for all directions $g=(h,0)$, there exists some $z \in \mathbb{R}_+^{n+p}$ such that $g=(h,0)$ is an infeasible direction at z.
b) Assume that $y \in P(\mathbb{R}_+^n)$ and suppose that the input set $L(y)$ has a nonempty interior in $\mathbb{R}_+^n$. If $h \in -\mathbb{R}_{++}$, then the input interior direction $g=(h,0)$ is feasible at $z=(x,y)$.

Proof: a) If $P(x)$ is a bounded set, then there exists $y \in \mathbb{R}_+^p$ such that $y \notin P(x)$. Now for all $\delta \in \mathbb{R}$, we have $(x,y)+\delta(h,0) \notin T$ and this ends the proof. b) Since $L(y)$ has a nonempty interior, there is some $u \in L(y) \cap \mathbb{R}_{++}^n$. Moreover, since $h \in \mathbb{R}_{++}^n$, there is some $\bar{\delta} \in \mathbb{R}$ such that $x+\bar{\delta}h \geq u$. Since the free disposal assumption holds, we deduce that $x+\bar{\delta}h \in L(y)$. This ends the proof. $\square$
To illustrate the a) part of this proposition, we cite a few empirical studies explicitly reporting the prevalence of this infeasibility problem in the case of the input-oriented Malmquist index.[9] Glass and McKillop (2000) mention for their sample of 84 UK building societies that 5, 6 and 6 observations (about 7%) encounter infeasibilities when comparing their distances to technologies situated in different time periods. Mukherjee, Ray and Miller (2001) report between 1% and 3.5% infeasibilities on a larger sample of 201 US commercial banks over a longer number of years (see their Tables 4-6).

The following corollary is immediate:

Corollary 3.11 *Let T be a production technology satisfying $T1-T4$. Moreover, assume that T has a nonempty interior, $p=1$ and constant returns to scale hold. For all $y \in \mathbb{R}_+$ if $h \in -\mathbb{R}_{++}^n$, then the input interior direction $g=(h,0)$ is feasible at $z=(x,y)$.*

Proof: Since T has a nonempty interior, then $L(y)$ has also a nonempty interior in $\mathbb{R}_+^n$. However, since $p=1$ for all $y \in \mathbb{R}_+$, $L(y) \neq \emptyset$ and this ends the proof. $\square$

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[8] We say that the output correspondence is bounded if there exists a compact $K \subset \mathbb{R}_+^p$ such that $P(x) \subset K$ for all $x \in \mathbb{R}_+^n$.

[9] Though we are unaware of articles reporting infeasibilities, the Luenberger input-oriented productivity indicator could suffer from the same problems.

This corollary explains that in the single output case imposing constant returns to scale and a full dimensional input direction vector are sufficient conditions for feasibility.

In the literature on the Malmquist productivity index, the impression is given that the infeasibility issue can be solved by simply imposing constant returns to scale on a non-parametric technology (see, e.g., Färe, Grosskopf and Lovell (1994)). However, the above propositions clearly demonstrate that the occurrence of infeasibilities in, for instance, the case of the input-oriented Malmquist index is not linked to a returns to scale hypothesis imposed on technology, but that it depends on the output direction vector being null and the input direction vector not being of full dimension. Furthermore, constant returns to scale in itself is never a sufficient condition to guarantee feasibility.

Thus, both the use of parametric and non-parametric technologies can generate infeasibilities when computing discrete time productivity indexes when the output direction vector is null and the input direction vector is not of full dimension.

3.3 Duality and Feasibility

One of the key results so far, proven in Proposition 3.2, is that if $k \neq 0$ then there is some $z \in \mathbb{R}^{n+p}_{+}$ such that the direction g is infeasible at z. Therefore, it is obvious that if $g \in (-\mathbb{R}^{n}_{++}) \times \mathbb{R}^{p}_{++}$, then there is some $z \in \mathbb{R}^{n+p}_{+}$ such that $D_T(z; g) = -\infty$. In this subsection we show, perhaps surprisingly, that this results does not hold true for the dual formulation of the directional distance function.

To show this, we introduce the *free disposal cone* that is defined as:

$$K = \mathbb{R}^{n}_{+} \times (-\mathbb{R}^{p}_{+}) \tag{3.2}$$

This cone is related to the free disposal assumption because T4 can be equivalently written as $(T + K) \cap \mathbb{R}^{n+p}_{+} = T$. Throughout this subsection this free disposal cone plays a crucial role.

The next main result establishes that if the line $\Delta(z, g)$ meets the addition of the technology and the free disposal cone $T + K$, then the dual directional distance function is well-defined.

Proposition 3.12 *Let T be a production technology satisfying T1-T5. For all $z \in \mathbb{R}^{n+p}_{+}$, if $\Delta(z, g) \cap (T + K) \neq \emptyset$, then:*

$$\bar{D}_T(z; g) > -\infty,$$

and:

$$\bar{D}_T(z; g) = \max\{\delta : z + \delta g \in T + K\}.$$

Moreover, there exist $(\bar{x}, \bar{y}) \in \mathbb{R}^{n+p}_{+}$ and $(\bar{w}, \bar{p}) \in \mathbb{R}^{n+p}_{+}$ with $\bar{p}.k - \bar{w}.h = 1$ such that:

$$\bar{D}_T(z; g) = \bar{p}.\bar{y} - \bar{w}.\bar{x} - \bar{p}.y + \bar{w}.x.$$

Proof: Let us denote $\gamma(z;g) = \sup\{\delta : z + \delta g \in T + K\}$. Since $\Delta(z,g) \cap (T+K) \neq \emptyset$, that is closed, $\gamma(z;g) > -\infty$ and $z + \gamma(z;g)g \in T + K$. For all convex $C \subset \mathbb{R}^{n+p}$, let us define the function $h_C : \mathbb{R}_+^{n+p} \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ defined as $h_C(w,p) = \sup\{p.y - w.x : (x,y) \in C\}$. From the convexity of T we deduce the convexity of $T + K$. Since $z + \gamma(z;g)g \in Bd(T+K)$, from the weak version of the convex separation theorem, we deduce that there exists $(w,p) \in \mathbb{R}^{n+p}$ such that:

$$p.(y + D(x,y;g).k) - p.(x + D(x,y;g).h) = h_{T+K}(w,p)$$

It is, however, a standard fact that $h_{T+K}(w,p) = h_T(w,p) + h_K(w,p)$ and since $h_K(w,p) = +\infty$ for all $(w,p) \notin \mathbb{R}_+^{n+p}$, we deduce that $(w,p) \in \mathbb{R}_+^{n+p}$. Moreover, since $h_T(w,p) = \Pi(w,p)$ and $h_K(w,p) = 0$, an elementary calculus show that:

$$\gamma(z;g) = \frac{\Pi(w,p) - p.y + w.x}{p.k - w.h}$$

Therefore, for all $(w',p') \in \mathbb{R}_+^{n+p}$ if $h_{T+K}(w',p') = \Pi(w',p') < +\infty$ then we have

$$\sup\{\delta : p'.(y + \delta k) - p'.(x + \delta h) \leq \Pi(w',p')\} \geq \frac{\Pi(w,p) - p.y + w.x}{p.k - w.h}$$

and normalizing, we deduce that

$$\gamma(z;g) = \min_{(w,p) \geq 0} \{\Pi(w,p) - p.y + w.x : p.k - w.h = 1\} > -\infty$$

Therefore, since the minimum is achieved, there is some $(\bar{w},\bar{p}) \in \mathbb{R}_+^{n+p}$ and such that $\Pi(\bar{w},\bar{p}) = (-w,p).\left(z + \gamma(z;g)g\right)$. But, since $z + \gamma(z;g)g \in T + K$, there is some $(\bar{x},\bar{y}) \in T$ such that $z + \gamma(z;g)g \in (\bar{x},\bar{y}) + K$, and consequently we have immediately $\Pi(\bar{w},\bar{p}) = \bar{p}.\bar{y} - \bar{w}.\bar{x}$. □

This has an immediate consequence: if all components of the direction vector are non-zero, then the dual directional distance function is well-defined. Otherwise, the dual directional distance function may well not solve the infeasibility problem.

Corollary 3.13 *Let T be a production technology satisfying T1-T5. Let $g \in (-\mathbb{R}_{++}^n) \times \mathbb{R}_{++}^p$ be an interior direction. For all $z \in \mathbb{R}_+^{n+p}$, we have:*

$$\bar{D}_T(z;g) > -\infty.$$

Proof: If $g \in (-\mathbb{R}_{++}^n) \times \mathbb{R}_{++}^p$, then there is some $\bar{\delta} \in \mathbb{R}_-$ such that $y + \bar{\delta}k \leq 0 \Longrightarrow z + \bar{\delta}g \in \{(0,0)\} + K \Longrightarrow z + \delta g \in T + K$. Therefore, $\Delta(z,g) \cap (T+K) \neq \emptyset$ and from Proposition 3.12 the result is established. □

This result can be related to Briec and Lesourd (1999) who showed that if $g = (-1^n, 1^p)$ then, for all $z \in T$, $\bar{D}_T(z;g)$ is the Tshebishev distance from z

to the weak efficient subset of T. Another corollary points out the difference between primal and dual directional distance functions for some infeasible directions.

Corollary 3.14 *Let T be a production technology satisfying T1-T-5. For all $z \in \mathbb{R}_+^{n+p}$, if $\Delta(z, g) \cap T = \emptyset$ and $\Delta(z, g) \cap (T + K) \neq \emptyset$, then:*

$$\bar{D}_T(z; g) > D_T(z; g) = -\infty.$$

This last result is illustrated in Figure 1. In Figure 1, we suppose that $g = (0, k)$. Therefore, for all price vectors $(\bar{w}, \bar{p}) \in \mathbb{R}_+^{n+p}$ such that $\bar{D}_T(z; g) = \Pi(\bar{w}, \bar{p}) - \bar{p}.y + \bar{w}.x$ with $\bar{p}.k - \bar{w}.h = 1$, we have $\Pi(\bar{w}, \bar{p}) - \bar{p}.y + \bar{w}.x = \bar{p}.(y + \gamma(z; g)k) - \bar{w}.(x + \gamma(z; g)h) - \bar{p}.y + \bar{w}.x = p.(y + \gamma(z; g)k) - \bar{p}.y = R(\bar{p}, x) - \bar{p}.y > 0$. Thus, there exist points and direction vectors for which the hyper-directional distance function may well be feasible, while the directional distance function is infeasible.

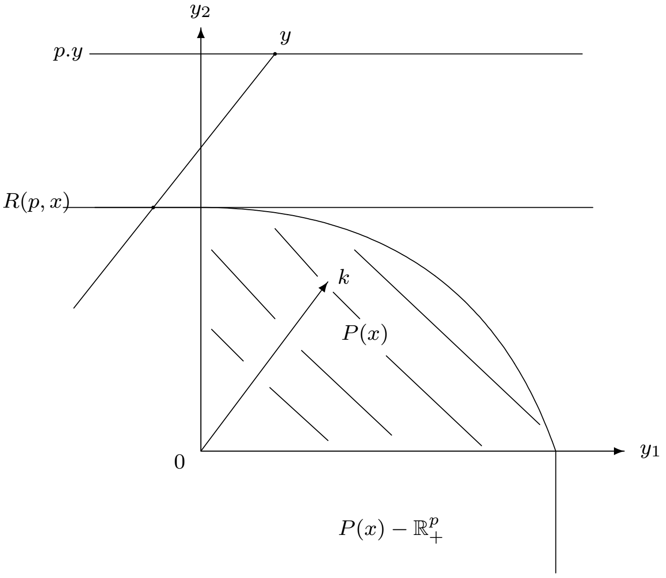

Figure 1 A case where $D_T(z; g) = -\infty$ and $\bar{D}_T(z; g) > -\infty$.

This same result is also illustrated by taking up again the earlier Example 3.3 and showing that its dual directional distance function is feasible.

Example 3.15 *Let us consider Example 3.3 where for $n = 1$ and $p = 2$, the production technology is $T = \{(x, y_1, y_2) \in \mathbb{R}_+^3 : y_1 + y_2 \leq x\}$. We have shown that the direction $g = (-1, 1, 1)$ is not feasible at point $z = (1, 0, 2)$ and thereby $D_T(z; g) = -\infty$. However, we have shown in Proposition 3.12 that the dual directional distance function is $\bar{D}_T(z; g) = \sup\{\delta : (1, 0, 2) + \delta(-1, 1, 1) \in T + K\}$. Let us determine a maximization program to compute this dual directional distance function. Since the output dimension is not constrained in $T + K$, we have:*

$$T + K = \{(x, y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R}^2 : y_1 + y_2 \leq x\}$$

Therefore, the constraints $0+\delta \geq 0$ *and* $2+\delta \geq 0$ *in system 3.1 should be suppressed in the maximization program to compute the dual directional distance function:*

$$\begin{array}{rcl} \max\delta & & \\ 1-\delta & \geq & 0 \\ 2+2\delta & \leq & 1-\delta \end{array} \tag{3.3}$$

We obtain $\bar{D}_T(z;g) = -1/3 > -\infty = D_T(z;g)$.

To complete the main result above we establish that if the condition $\Delta(z,g) \cap (T+K) \neq \emptyset$ does not hold, then the dual directional distance function is infeasible ($\bar{D}_T(z;g) = -\infty$).

Proposition 3.16 *Let* T *be a production technology satisfying T1-T5. For all* $z \in \mathbb{R}_+^{n+p}$, *if* $\Delta(z,g) \cap (T+K) = \emptyset$, *then:*

$$\bar{D}_T(z;g) = -\infty.$$

Proof: If $\Delta(z,g) \cap (T+K) = \emptyset$, then there are two subsets $I = \{i \in \{1\cdots n\} : h_i = 0\}$ and $J = \{j \in \{1\cdots p\} : k_j = 0\}$ such that $I \cup J \neq \emptyset$. For all positive integers m, let us define the direction $g^m = (h^m, k^m)$ such that:

$$h_i^m = \begin{cases} h_i & \text{if } i \notin I \\ \frac{1}{m} & \text{if } i \in I \end{cases} \text{ and } k_j^m = \begin{cases} k_j & \text{if } j \notin J \\ \frac{1}{m} & \text{if } j \in J \end{cases}$$

Since $g^m \in (-\mathbb{R}_{++}^n) \times \mathbb{R}_{++}^p$, we deduce that $\Delta(z,g^m) \cap (T+K) \neq \emptyset$. Let $\gamma(z;g^m) = \sup\{\delta : z+\delta g^m \in T+K\}$. Let us prove that $\lim_{m\longrightarrow+\infty}\gamma(z;g^m) = -\infty$. Assume the contrary and let us show a contradiction. Since $\Delta(z,g) \cap (T+K) = \emptyset$, $z \notin T$. Therefore, $\gamma(z;g^m) \leq 0$ for all $m \in \mathbb{N}\backslash\{0\}$. Suppose that there is a compact $W \subset \mathbb{R}^{n+p}$ such that $w^m = z + \gamma(z;g^m)g^m \in W$ for all positive integer m. Since W is compact there is some subsequence $\{m_l\}_{l\in\mathbb{N}}$ such that $\lim_{l\longrightarrow\infty} w^{m_l} = w \in W$. However, $\lim_{l\longrightarrow\infty} g^{m_l} = g$. Consequently, there is some $\bar{\gamma}$ such that $\lim_{l\longrightarrow\infty}\gamma(z;g^{m_l}) = \bar{\gamma}$, and since $T+K$ is closed $w = z+\bar{\gamma}g \in T+K$. This is a contradiction because of the assumption $\Delta(z,g) \cap (T+K) = \emptyset$. Consequently, $\lim_{m\longrightarrow+\infty}\|z+\gamma(z;g^m)g^m\| = +\infty$ and since $\lim_{m\longrightarrow\infty} g^m = g$ we deduce that $\lim_{m\longrightarrow\infty}\gamma(z;g^m) = -\infty$. Now, for all $m \in \mathbb{N}$, since:

$$\left\{(w,p) \in \mathbb{R}_+^{n+p} : p.k - wh = 1\right\} \supset \left\{(w,p) \in \mathbb{R}_+^{n+p} : p.k^m - w.h^m = 1\right\},$$

we deduce that for all $m \in \mathbb{N}\backslash\{0\}$ we deduce that:

$$\begin{aligned} &\inf_{(w,p)\geq 0}\{\Pi(w,p) - p.y + w.x : p.k - w.h = 1\} \\ &\leq \min_{(w,p)\geq 0}\{\Pi(w,p) - p.y + w.x : p.k^m - w.h^m = 1\} = \gamma(z;g^m). \end{aligned}$$

Therefore, since $\lim_{m\longrightarrow\infty}\gamma(z;g^m) = -\infty$, we obtain:

$$\bar{D}_T(z;g) = -\infty. \quad \square$$

To conclude this discussion, we establish a final result indicating that the feasibility of the dual directional distance function is a necessary and sufficient condition to conclude that the intersection of a line with the technology extended by the free disposal cone is non-empty.

Theorem 3.17 *Let T be a production technology satisfying T1-T5. For all $z \in \mathbb{R}_+^{n+p}$,*

$$\Delta(z, g) \cap (T + K) = \emptyset \iff \bar{D}_T(z; g) = -\infty.$$

3.4 Existence of Feasible Directions

This subsection sets to determine the conditions for the existence of a feasible direction $\tilde{g}(z)$ at each point z in the non-negative Euclidean orthant. It turns out that the required necessary and sufficient conditions are very restrictive. For convenience, we use the following decomposition of the direction vector $\tilde{g}(z) = (\tilde{h}(z), \tilde{k}(z))$. More specifically, suppose that the direction vector is given by some function $\tilde{g} : \mathbb{R}_+^{n+p} \longrightarrow (-\mathbb{R}_+^n) \times \mathbb{R}_+^n$ termed the direction function. This direction function is defined as:

$$\tilde{g}(z) = (\tilde{h}(z), \tilde{k}(z)). \tag{3.4}$$

Proposition 3.18 *Assume that $p \geq 2$. Then, the two following conditions are equivalent:*
i) For all production technologies satisfying T1-T4 and all $z \in \mathbb{R}_+^{n+p}$, $\Delta(z, \tilde{g}(z)) \cap T \neq \emptyset$
ii) $\tilde{g}$ has the form $\tilde{g}(z) = (\tilde{h}(z), cy)$ where $c \in \mathbb{R}_{++}$.

Proof: Assume that ii) does not hold. Let $\bar{\delta} = \inf\{\delta : y + \delta\tilde{k}(z) \geq 0\}$. Since ii) does not hold and $p \geq 2$, there is some $j \in \{1 \cdots p\}$ such that $y_j + \bar{\delta}\tilde{k}_j(z) > 0$. Let T be an arbitrary production technology satisfying $T1 - T4$ such that $y + \bar{\delta}\tilde{k}(z) \in P(\mathbb{R}_+^n)$. This means that $y + \bar{\delta}\tilde{k}(z)$ can be produced by some input vector. Now let

$$H_j = \left\{(u, v) \in \mathbb{R}_+^{n+p} : v_j \leq \frac{1}{2}\left(y_j + \bar{\delta}\tilde{h}_j(z)\right)\right\}.$$

It is easy to check that $T \cap H_j$ satisfies $T1 - T4$ and $(x + \bar{\delta}\tilde{h}(z), y + \bar{\delta}k(z)) \notin T \cap H_j$. Consequently, $\Delta(z, \tilde{g}(z)) \cap (T \cap H_j) = \emptyset$ and this contradicts i). Thus i)$\Longrightarrow$ ii). Conversely, if ii) holds for $\bar{\delta} = (-1 + \frac{1}{c})$, then $y + \bar{\delta}cy = 0$, and since $x + \bar{\delta}\tilde{h}(z) \geq x$ and $(0, 0) \in T$, we deduce that $(x + \bar{\delta}\tilde{h}(z), y + \bar{\delta}z) = (x + \bar{\delta}\tilde{h}(z), 0) \in T$. Thus, $\Delta(z, \tilde{g}(z)) \cap T \neq \emptyset$ and i) holds. $\square$

Thus, when the direction vector is interior and strictly proportional in all output dimensions in the technology (and $p \geq 2$), then the directional distance function is always feasible. The following corollary is an immediate consequence.

Corollary 3.19 *For all production technology satisfying T1-T4 and all* $z \in \mathbb{R}_+^{n+p}$*, if the direction function has the form* $\tilde{g}(z) = (\tilde{h}(z), cy)$ *where* $c \in \mathbb{R}_{++}$*, then*

$$D_T(z; \tilde{g}(z)) > -\infty.$$

The above conditions underscore the importance of imposing minimal restrictions on the output direction to guarantee feasibility.

The next result establishes necessary and sufficient conditions in the case of an input-oriented direction vector. It turns out that if the output direction vector equals zero and an output vector is attainable from an input vector, then a necessary and sufficient condition for the directional distance function to be feasible is that the direction vector is input interior for all production vectors z.

Proposition 3.20 *Suppose that* $\tilde{g} : \mathbb{R}_+^{n+p} \longrightarrow (-\mathbb{R}_+^n) \times \{0\}^p$ *is a direction function. Let* $(x, y) \in \mathbb{R}_+^{n+p}$ *and suppose that* $y \in P(\mathbb{R}_+^n)$*. The two following conditions are equivalent:*
i) For all production technologies satisfying T1-T4 and all $z \in \mathbb{R}_+^{n+p}$*,* $\Delta(z, \tilde{g}(z)) \cap T \neq \emptyset$
ii) $\tilde{g}$ *has the form* $\tilde{g}(z) = (\tilde{h}(z), 0)$ *where* $\tilde{h}(\mathbb{R}_+^{n+p}) \subset -\mathbb{R}_{++}^n$*.*

Proof: From Proposition 3.7 it is clear if ii) does not hold true, then i) does not hold true. Therefore, i) $\Longrightarrow$ ii). Let us prove that ii) $\Longrightarrow$ i). Since $y \in P(\mathbb{R}_+^n)$, there is some $\bar{x} \in \mathbb{R}_+^n$ such that $y \in P(\bar{x})$, thus $(\bar{x}, y) \in T$ and $y \in L(\bar{x})$. Now, since $\tilde{h}(\mathbb{R}_+^{n+p}) \subset -\mathbb{R}_{++}^n$, there exists some $\bar{\delta} < 0$ such that $x + \bar{\delta}\tilde{h}(z) > \bar{x}$. Therefore, from the strong disposal assumption $(x + \bar{\delta}\tilde{h}(z), y) \in T$ and consequently since $\tilde{k}(z) = 0$, $(x, y) + \bar{\delta}\tilde{g}(z) \in T$. This ends the proof.□

This excludes all sub-vector orientations in the inputs when the output direction vector is null. Ouellette and Vierstraete (2004) are an example of a study reporting infeasibilities (in particular, 1 out of 15 observations) for a sub-vector input-oriented Malmquist productivity index.

4 Conclusions

This paper has verified in detail under which conditions the directional distance function, the most general distance function introduced in the literature so far, may not achieve its distance in the general case where a point need not be part of technology and where the direction vector can take any value. In section 3 we demonstrated a perfectly general result that in the case of more than two output dimensions and non-null output direction vector, the directional distance function may be infeasible. In addition to a series of more specific infeasibility results, it has been demonstrated that the hyper-directional distance function, the dual version of the standard directional distance function, is always feasible for interior directions.

Apart from reporting any eventual infeasibilities, this contribution shows that there is no easy solution in general. While a general solution to the problem exists under rather stringent conditions, it remains the case that in a variety of circumstances the problem of infeasibilities cannot be avoided irrespective of the estimation method used for technology. Also, the current results can be partly interpreted as providing support for the proportional directional distance function, whereby the direction vector equals the evaluated observation. Consequently, since in general the directional distance function may not be well-defined, the axiom of determinateness in index theory should be firmly rejected.

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