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S-DISPOSABILITY ASSUMPTION AND DUALITY BETWEEN TECHNOLOGY AND COST FUNCTION

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Abstract

The purpose of this contribution is to define a relaxed notion of disposability for technology. This S-disposal assumption leads to a duality result between a general input directional distance function and the cost function that is weaker than the ones hitherto established in the literature. Furthermore, this new axiom is also helpful to detect and measure more severe forms of congestion in production than the ones that could hitherto be evaluated.

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1 Introduction

Congestion -here intuitively defined as production with negative marginal product- is mainly mentioned as a theoretical curiosity in production theory. Indeed, when discussing the notions of average and marginal productivity, for instance, the use of the qualifications "irrational" to certain of the so-called "stages of production" clearly points out the low probability attributed to congestion occurring in practice (e.g., Ferguson (1969: 66-79)). What seems often ignored is the theory-dependency of observations: to detect a phenomenon, one must have a theoretical construction allowing for the phenomenon to be observed.

In applied production analysis, many functional forms employed cannot detect congestion at all. For instance, the Cobb Douglas specification imposes positive marginal productivity along the isoquant throughout the input

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space. Furthermore, the widespread use of flexible functional forms created a practice of imposing curvature globally, while monotonicity is only imposed locally (to maintain flexibility) or not at all. Barnett (2002) describes and illustrates some of the available evidence indicating that imposing curvature solely can actually induce violations of monotonicity. Without the satisfaction of both curvature and monotonicity, the standard second-order conditions for optimising behaviour fail and duality theory breaks down. Apart from these flexible functional forms, there is -to our knowledge- only the ray or weakly disposable production function, which is basically a generalisation of the variable elasticity of substitution function, that is capable to identify congestion. However, it has rarely been applied empirically and the few existing empirical studies focused mainly on disembodied technical change that widens productive factor combinations rather than detecting congestion per se (Färe and Jansson (1974)). In non-parametric production theory multi-output ray (weakly) and free (strongly) disposable technologies have been employed to distinguish between technical inefficiency on the one hand, understood as production below a frontier technology, and congestion on the other hand, interpreted as a particular severe form of technical inefficiency whereby additional inputs actually decrease outputs (see Färe and Grosskopf (1983)).

Though congestion is still often neglected in empirical analysis, there are several examples indicating that it may even be the most important source of poor performance (see, e.g., Byrnes et al. (1988)). Prominent examples of congestion phenomena are traffic congestion, agricultural output losses due to excessive use of fertilisers or climatological circumstances, output reduction due to featherbedding or rigid work rules, etc. (Färe and Grosskopf (1983)).

While duality theory between the cost function and the input distance function is traditionally established imposing strong disposability of inputs (e.g., Shephard (1970) or Luenberger (1995)), also a weaker duality result between the cost function and the ray (or weakly) disposable input distance function is available in the literature whereby some (but not all) prices are allowed to be negative (see, e.g., Shephard (1974)). To highlight the importance of the ray disposal axiom in axiomatic production theory, notice that ray (weak) disposal of inputs is a necessary and sufficient condition for the input distance function to characterize technology (Färe and Primont (1995)).[1]

The main purpose of this contribution is to establish a more general duality result based on a relaxed disposability assumption. This research is driven by a double motivation. First, we consider the axiom of ray disposability of inputs intuitively unappealing, since it amounts to assuming that inputs can be disposed off along a ray without limitation. Therefore, we suggest to replace this ray disposability assumption with a weaker S-disposal assumption

[1] In a similar way, the ray disposability axiom in the outputs is a necessary and sufficient condition for the output distance function to characterize technology.

that -inspired by Lau (1974: 182)- essentially makes the strong disposability assumption a local rather than a global property. In this view, since only variables with values within a certain domain are relevant it suffices to maintain the monotonicity property within a prescribed domain as dictated by sample information. Therefore, the S-disposal assumption can model more general forms of congestion than the ray disposability assumption, including the case of limits on the ray disposal of all inputs.

Second, the ray disposability assumption has so far never been employed to model non-convex production models. Since on the one hand ray disposal models have gained some popularity to explicitly model the trade-offs between good and bad outputs for the environment (see, e.g., Coggins and Swinton (1996)) and since on the other hand it has recently become apparent that the acceptance/rejection of convexity may well constitute a main cleavage between economics and ecology (see, e.g., Dasgupta and Mähler (2003)), it seems to be important to have ways of modeling congestion that do not impose convexity.[2] This is in principle possible with the S-disposal assumption.

This paper then proposes a new way to model congestion using a relaxed disposability assumption that allows defining technologies enveloping the data tighter than hitherto possible. The main reason for this methodological innovation is to reveal any congestion in production processes compatible with a minimal set of assumptions. One main limitation is that we focus on congestion in the input space solely. Therefore, we concentrate on the input distance functions and its dual relation with the cost function to characterise congestion. Generalisations to congestion phenomena in the outputs space or to the input and output space are relatively straightforward, but are deferred to a later contribution. Another limitation is that we ignore the consequences of relaxing the strong disposal assumption for general equilibrium theory. Indeed, it is rather well-known that the free disposal assumption cannot be dropped or even be relaxed without risking that equilibria may fail to exist in non-convex (e.g., increasing returns to scale) economies (see, for instance, the recent contribution of Salchow (2006)).

This paper unfolds as follows. Section 2 contains preliminary material on technologies, their subsets and underlying axioms. It also presents the new disposability axioms on technologies. Looking from a dual viewpoint, we focus on the fact that negative relative prices are linked to the congestion notion. Section 3 develops the notion of input directional distance functions on congested technologies and establishes the main duality result between the input directional distance function and a cost function allowing for negative prices. Furthermore, we show how to detect a lack of disposability and outline a measure of congestion. A final section concludes and offers directions for future research.

[2]A criticism on convexity in production theory (and economics in general) based on the importance of indivisibilities is developed in Scarf (1986).

2 Technology: Assumptions and Definitions

2.1 Technology Based upon Traditional Assumptions

First, we define the notation used in this article. Let $\mathbb{R}_+^M$ be the non-negative Euclidean M-orthant; for $x, u \in \mathbb{R}_+^M$ we denote $x \leq u \iff x_m \leq u_m$ $\forall m \in \{1, ..., M\}$.

A production technology transforming inputs $x = (x_1, ..., x_M)$ into outputs $y = (y_1, ..., y_N) \in \mathbb{R}_+^N$ can be characterized by the input correspondence $L : \mathbb{R}_+^M \longrightarrow 2^{\mathbb{R}_+^M}$ where $L(y)$ is the set of all input vectors that yield at least y:

$$L(y) = \{x : x \text{ can produce } y\}. \tag{2.1}$$

This input correspondence can be completely characterized by the input distance function $D_i : \mathbb{R}_+^{M+N} \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ defined by:

$$D_i(x, y) = \begin{cases} \sup\{\lambda : x/\lambda \in L(y)\} & \text{if } R_x \cap L(y) \neq \emptyset \\ -\infty & \text{otherwise} \end{cases} \tag{2.2}$$

where $R_x = \{tx : t \geq 0\}$. As developed underneath, functions based upon this input distance function serve an important role in the measurement of congestion in its various forms.

Throughout this paper, we assume the input correspondence satisfies the following regularity properties (see Färe and Primont (1995), McFadden (1978), or Shephard (1970)):

$L1$: $\forall y \geq 0$ with $y \neq 0$, $0 \notin L(y)$ and $L(0) = \mathbb{R}_+^M$.
$L2$: Let $\{y_n\}_{n \in \mathbb{N}}$ be a sequence such that $\lim_{n \to \infty} \|y_n\| = \infty$, then $\bigcap_{n \in N} L(y_n) \neq \emptyset$.
$L3$: $L(y)$ is closed $\forall y \in \mathbb{R}_+^N$.

In addition to the axioms of no free lunch and the possibility of inaction, as well as the boundedness and closedness of the inputs set, there are three other assumptions that we sometimes invoke on the input correspondence:

$L4$: $L(y)$ is a convex set $\forall y \in \mathbb{R}_+^N$.
$L5$: If $x \in L(y)$ then $\lambda x \in L(y)$, $\forall \lambda \geq 1$.
$L6$: $\forall x \in L(y), u \geq x \Rightarrow u \in L(y)$.

Assumption L4 postulates convexity of the input correspondence. This is useful to provide a dual interpretation through the cost function and in empirical applications of, for instance, non-parametric technologies (e.g., Varian (1984)). Assumption $L5$ postulates ray (or weak) disposability of the inputs, while axiom $L6$ imposes the more traditional assumption of strong (or free) disposal of inputs. A convex technology satisfying L5 but failing $L6$ is congested in the sense of Färe and Grosskopf (1983). Notice that $L4$

is not indispensable, since there exist non-parametric non-convex technologies solely based upon free disposal ($L6$) (e.g., Briec, Kerstens and Vanden Eeckaut (2004)).

To measure efficiency, it is convenient to distinguish between certain subsets of the input set $L(y)$. In particular, two subsets denoting production units on the boundary prove useful. The efficient subset is defined by:

$$E(y) = \{x \in L(y) : u \leq x \text{ and } u \neq x \Rightarrow u \notin L(y)\}. \tag{2.3}$$

The weak efficient subset is written as:

$$W(y) = \{x \in L(y) : u < x \Rightarrow u \notin L(y)\}. \tag{2.4}$$

2.2 A Relaxed Disposability Assumption

2.2.1 Congestion, S-Disposability assumption and S-congestion

We start out with a more precise definition of congestion. Transposing Färe and Svensson (1980) from the single output to the multiple output case, Färe and Grosskopf (1983: 264) define monotone output-limitational (MOL) congestion as follows:

Definition 2.2.1 *Input correspondence $L(y)$ is MOL-congested if for some $y \geq 0$ and $x \in L(y)$, $\exists x' \geq x$ such that $x' \notin L(y)$.*

This means that a technology is MOL-congested if it fails the free disposal assumption. For instance, a ray disposable technology is MOL-congested (Färe and Grosskopf (1983)).

We first define the notation used in this contribution. Let $I \subset \{1, ..., M\}$, we introduce the following symbol:

$$x \leq^I u \iff \begin{cases} x_i \geq u_i & \text{if } i \in I \\ x_i \leq u_i & \text{else} \end{cases} \tag{2.5}$$

Moreover:

$$x <^I u \iff \begin{cases} x_i > u_i & \text{if } i \in I \\ x_i < u_i & \text{else} \end{cases} \tag{2.6}$$

Of course, if $-x \leq^I -u$ we denote $x \geq^I u$.

We are now able to define a new kind of disposability assumption for the inputs. Denote by $2^{\{1,...,M\}}$ the set of all partitions of $\{1, ..., M\}$. Remark that, by definition, $\emptyset \in 2^{\{1,...,M\}}$.

Definition 2.2.2 *Let L be an input correspondence satisfying L1-L3. Let $S \in 2^{\{1,...,M\}}$. The input-correspondence $L(y)$ satisfies S-disposal assumption if for all sets of input vectors $\left\{x^I\right\}_{I \in S} \subset L(y)$, $x \geq^I x^I$ for any $I \in S$ implies that $x \in L(y)$.*

Notice that if $S = \emptyset$, then we retrieve the standard vector inequality and the S-disposal assumption reduces to the standard free disposability assumption.

Definition 2.2.3 *Let L be an input correspondence satisfying L1-L3. Let $S \in 2^{\{1,\ldots,M\}}$. $L(y)$ satisfies a minimal S-disposability assumption if:*
a) $L(y)$ satisfies the S-disposal assumption, and
b) $\nexists\ S' \subset S$ with $S' \neq S$ such that $L(y)$ satisfies the S'-disposal assumption.

Definitions 2.2.2 and 2.2.3 can be illustrated in Figures 2.2.1.1 and 2.2.1.2. In Figure 2.2.1.1, we have $S = \{\emptyset, \{1\}\}$. For any x, if there is some $x^{\emptyset}$ that classically dominates x and some x^1 that "$\{1\}$-dominates" x, then $x \in L(y)$. For a given configuration of observations, this serves to construct an input set where wasting the first input implies an additional opportunity cost in terms of the second input dimension. However, the reverse dependency between input dimensions does not hold.

Basically, the free disposability assumption is weakened by combining it with a particular partial reversion of a free disposability assumption. Another way of interpreting this new definition is to say that we try to reformulate the traditional strong input disposability assumption as a local rather than a global property (following the concerns expressed by Lau (1974)).

Figure 2.2.1.2 illustrates the case where $S = \{\emptyset, \{1, 2\}\}$: it is easy to see that a bounded set may satisfy the S-disposal assumption. This also serves to illustrate the main distinction with current weak disposability assumptions where any given input vector can be expanded along a ray through the origin. Consequently, there is no upper bound to wasting inputs, which seems a rather implausible assumption. Continuing the same logic, one could eventually also add the S-disposal assumption on the first and second input dimensions separately. The more input dimensions are subjected to these particular, partial reversions of free disposability defined above, the more the traditional free disposability assumption gets weakened. Indeed, Definitions 2.2.2 and 2.2.3 imply that the larger the collection S is the more difficult one can dispose off inputs.

But, not only can one use Definition 2.2.3 to account for cases where there is a simultaneous lack of free disposability in all dimensions, it is also possible to define this lack independently in several directions. In figure 2.2.1.3, there is a lack of disposability in x_1 and x_2, but not in both dimensions simultaneously. Thus, in this case $S = \{\emptyset, \{1\}, \{2\}\}$.

Let us introduce the following convex cone:

$$K^I = \left\{ x \in \mathbb{R}^M : x \geq^I 0 \right\}. \tag{2.7}$$

For simplicity, we denote the non-negative Euclidean orthant as follows:

$$K^{\emptyset} = \mathbb{R}^M_+. \tag{2.8}$$

To study this new disposal assumption from a dual standpoint, we introduce the cost function $C : \mathbb{R}^M \times \mathbb{R}^N_+ \Rightarrow \mathbb{R} \cup \{-\infty\}$ defined by:

$$C(p, y) = \inf_x \{p.x : x \in L(y)\}. \tag{2.9}$$

Notice that this definition allows to take into account negative prices which are specifically linked to congested technologies.

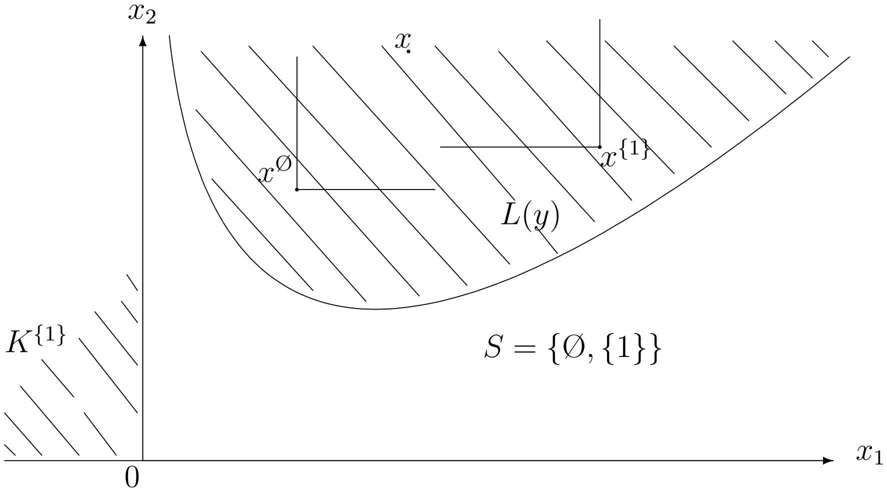

Figure 2.2.1.1: The case $S = \{\emptyset, \{1\}\}$ on a convex input set.

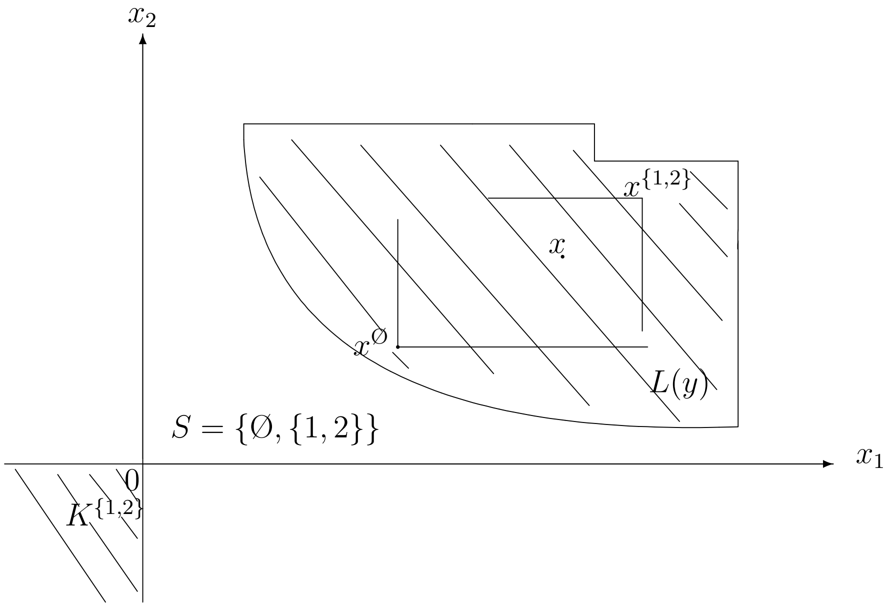

Figure 2.2.1.2: The case $S = \{\emptyset, \{1, 2\}\}$ on a convex input set.

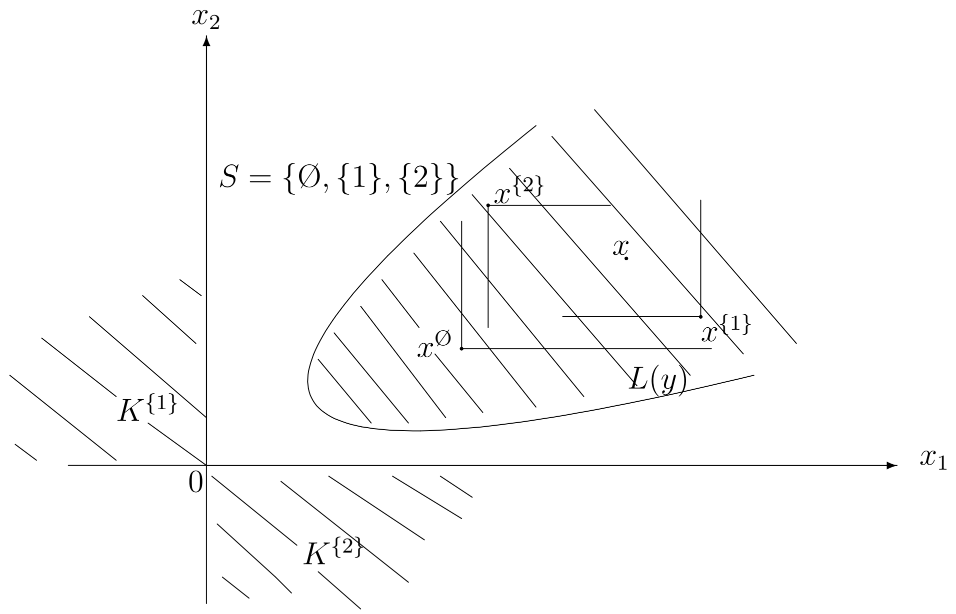

Figure 2.2.1.3: The case $S = \{\emptyset, \{1\}, \{2\}\}$.

In the following proposition, we study the properties of the S-disposal assumption.

Proposition 2.2.4 *Let L be an input correspondence satisfying L1-L3. We have the following properties:*
a) Let S and S' two collections of subsets of $\{1, ..., M\}$ such that $S' \subset S$. If $L(y)$ satisfies the S-disposal assumption, then it also satisfies the S'-disposal assumption.
b) $L(y)$ satisfies the S-disposal assumption if and only if:

$$L(y) = \bigcap_{I \in S} \left(L(y) + K^I \right).$$

Part a states that if an input set satisfies S-disposal of a certain dimensionality, then the same technology is compatible with S-disposal for any proper subset of the initial partition S. Part b characterizes an S-disposal input set in terms of an intersection of the convex cones in (2.7). Remark that when $S = \emptyset$, then the new disposal assumption reduces to the traditional free disposability assumption: $L(y) = L(y) + \mathbb{R}^M_+$.

The following proposition extends the results of Proposition 2.2.4 to a convex input correspondence. In particular, we provide a dual characterisation of the S-disposability notion.

Proposition 2.2.5 *Let L be an input correspondence satisfying L1-L3. Assume moreover that L4 holds. We have the following properties:*

a) $L(y)$ satisfies the S-disposal disposal assumption if and only if

$$L(y)=\left\{x\in\mathbb{R}^M : p.x\geq C(p,y), p\in\bigcup_{I\in S}K^I\right\}.$$

b) There exists a collection S that contains $\emptyset$ such that $L(y)$ satisfies a minimal S-disposal assumption.
c) Assume that $L(y)$ satisfies the S-disposal assumption. The S-disposability of $L(y)$ is minimal, if and only if for any $J\in S$ there exists some price vector $p\in K^J\cap\left(\mathbb{R}^M\backslash\bigcup_{I\in S\backslash\{J\}}K^I\right)$ such that $C(p,y)>-\infty$.

Intuitively stated, a convex input set satisfying S-disposal can be enveloped by a cost function for proper prices. Or, more precisely, if one defines a minimal S-disposal input set, then a support function can be defined with negative prices corresponding to the subset of all congesting input dimensions. This result constitutes the basis for the duality result developed in section 3.

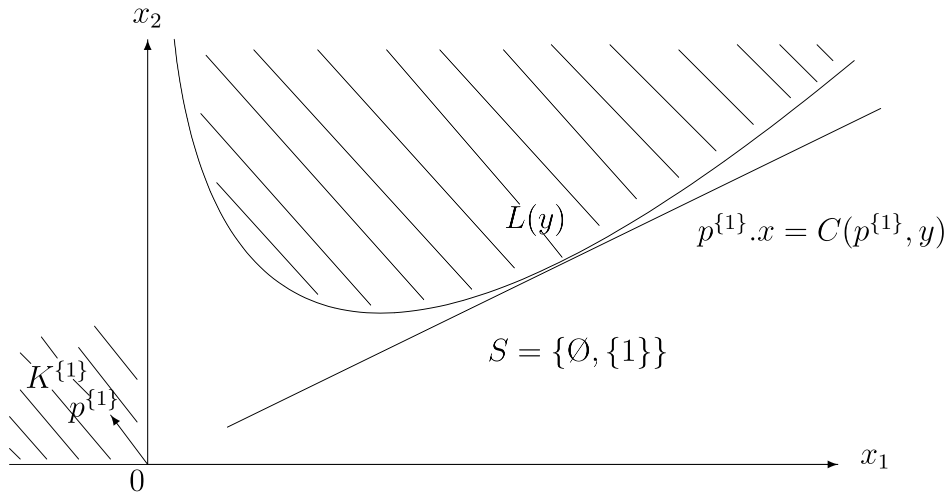

Figure 2.2.1.4: A minimal S-disposal assumption with $S=\{\emptyset,\{1\}\}$.

Figure 2.2.1.4 illustrates Proposition 2.2.5.c when $S=\{\emptyset,\{1\}\}$. We have $p^{\{1\}}\in K^{\{1\}}\cap\left(\mathbb{R}^M\backslash K^{\emptyset}\right)=K^{\{1\}}\cap\left(\mathbb{R}^M\backslash\mathbb{R}^M_+\right)$. We are now ready to define a new, more general congestion notion:

Definition 2.2.6 *Let L be an input correspondence satisfying L1-L3 and let S be a collection of subsets in $\{1,...,M\}$ that contains $\emptyset$. $L(y)$ is said to be S-congested if it fails the S-disposal assumption.*

Definition 2.2.6 provides a strict definition of S-congestion by assuming that there does not exist a stronger disposal assumption holding over the input correspondence. In particular, this means that a S-congested technology is such that:

$$L(y) \neq L^{\emptyset}(y) = L(y) + \mathbb{R}^M_+. \tag{2.10}$$

This can be be viewed as a transposition of the earlier definition of MOL-congestion in terms of S-disposal assumption. This facilitates comparisons among concepts. The next result establishes a characterization of S-congested technologies.

Proposition 2.2.7 *Let L be an input correspondence that satisfies L1-L3. Let S be a collection of subsets in $\{1, ..., M\}$ that contains $\emptyset$. We have the following properties:*
a) Assume that $L(y)$ is S-congested. For any $S' \subset S$ with $S' \neq \emptyset$, L is S'-congested.
b) Assume that $L(y)$ satisfies a minimal S-disposability assumption, then for any $S' \subset S$ with $S' \neq S$, $L(y)$ is S'-congested.
c) Assume that L4 holds. $L(y)$ is S-congested if and only if there exists $J \notin S$, and some price vector $p^J \in K^J \cap \left(\mathbb{R}^M \setminus \bigcup_{I \in S \setminus \{J\}} K^I\right)$ such that $C(p^J, y) > -\infty$.

Parts a and b of Proposition 2.2.7 state that if an input set is S-congested in terms of a certain dimensionality, then the same technology is S-congested for any proper subset of the initial partition S. This explains why congestion may remain unnoticed: assuming congestion is initially present in some dimensions of the true technology, then it is always possible to ignore congestion in some of these dimensions until eventually no congestion appears at all.

2.2.2 Boundaries and Bounds on S-congested Technologies

It remains an open question how to detect congestion from the structure of the input correspondence? To answer this question, it is useful to introduce the concept of a congestion frontier. Therefore, the following definition identifies a subset that is not efficient, but that is a part of the boundary of a congested input correspondence.

Definition 2.2.8 *Let L be an input correspondence satisfying L1-L3 and let $I \subset \{1, ..., M\}$.*
We call I-congested boundary the subset:

$$E^I(y) = \{x \in L(y) : u \leq^I x \text{ and } u \neq x \Rightarrow u \notin L(y)\}.$$

We call I-weakly congested boundary the subset:

$$W^I(y) = \{x \in L(y) : u <^I x \Rightarrow u \notin L(y)\}.$$

Let S be a collection of subsets of $\{1, ..., M\}$, we call respectively S-congested boundary and weakly S-congested boundary the union of subsets:

$$E^S(y) = \bigcup_{I \in S} E^I(y) \text{ and } W^S(y) = \bigcup_{I \in S} W^I(y).$$

Proposition 2.2.9 *Let L be an input correspondence satisfying L1-L3 and S be a collection of subsets of $\{1, ..., M\}$ that contains $\emptyset$.*
a) The subsets $E^S(y)$ and $W^S(y)$ are closed.
b) The correspondence set $L(y)$ is S-congested if and only if there exists some $J \notin S$ such that the subset $E^J(y)$ is non-empty.
c) The correspondence set $L(y)$ satisfies a minimal S-disposal assumption if and only if for any $J \in S$ the subset $E^J(y)$ is non-empty.
d) Assume that L4 holds. The correspondence set $L(y)$ is S-congested if and only if there exists some $J \notin S$, $x^J \in W^J(y)$ and some price vector $p^J \in K^J \cap \left(\mathbb{R}^M \backslash \bigcup_{I \in S \backslash \{J\}} K^I\right)$ such that $C(p^J, y) > -\infty$ and $p^J.x^J = C(p^J, y)$.

In propositions 2.2.5, 2.2.7 and 2.2.9 we have developed some connections between the S-congestion concept and the cost function. Obviously, when the free disposability assumption holds, then $C(p, y) > -\infty \iff p \geq 0$. However, the S-disposal assumption condition $p^J \in K^J \cap \left(\mathbb{R}^M \backslash \bigcup_{I \in S \backslash \{J\}} K^I\right)$ does not warrant that $C(p^J, y) > -\infty$. In fact, to obtain a similar property on the cost function, we introduce the S-bounded concepts. When the usual free disposability rule holds, since $L(y) \subset \mathbb{R}^M_+$, then the input correspondence is $\emptyset$-bounded.

Definition 2.2.10 *Let L be an input correspondence satisfying L1-L3 and S be a collection of subsets of $\{1, ..., M\}$ that contains $\emptyset$. The subset $L(y)$ is S-bounded if for any $I \in S$ there exists some $\bar{x}^I \leq^I x, \forall x \in \mathbb{R}^M_+$.*

Obviously, an input set that satisfies the usual free disposal assumption is $\emptyset$-bounded, with $\bar{x}^{\emptyset} = 0$. We show in Proposition 2.2.11 below that the above Definition 2.2.8 is of particular interest in the context of defining empirical specifications (e.g., non-parametric) of technologies.

Proposition 2.2.11 *Let L an input correspondence that satisfies L1-L3. Let S a collection of subsets in $\{1, ..., M\}$ that contains $\emptyset$. We have the following properties:*
a) If $L(y)$ is S-bounded and $\bigcup_{I \in S} I = \{1, ..., M\}$, then $L(y)$ is compact.
b) If $L(y)$ is S-bounded, then for any $S' \subset S$, with $S' \neq S$, it is S'-congested.
c) Assume that L4 holds, if $L(y)$ is S-bounded, then there exists some $J \in S$ and $p^J \in K^J \cap \left(\mathbb{R}^M \backslash \bigcup_{I \in S \backslash \{J\}} K^I\right)$ such that $C(p^J, y) > -\infty$.

3 A New Duality Result between Technology and Cost Function Based on S-Disposability

Recently, Luenberger (1992) introduced the so-called benefit function in consumer theory. Chambers, Chung and Färe (1996) have transposed this measure in the context of production theory by defining the input directional distance function. This input directional distance function provides a useful tool in efficiency and productivity measurement because it generalises the traditional input distance function, introduced above, and thus also the radial Debreu-Farrell efficiency measure (see Briec (1997)). The reason is that the input directional distance function is a special case of the directional distance function that itself is dual to the profit function (see Chambers, Chung and Färe (1998)). Therefore, the use of directional distance functions offers the most general framework, allowing easily to extend our proposals to define congestion in the output space or in the input-output space.

3.1 Directional Distance Function and Cost Function on S-Congested Technologies: A Duality Result

The input directional distance function $D_L : \mathbb{R}_+^{M+N} \times \mathbb{R}_+^M \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is defined by:

$$D_L(x, y; g) = \begin{cases} \sup\{\delta : x - \delta g \in L(y)\} & \text{if } x - \delta g \in L(y) \text{ for some } \delta \in \mathbb{R} \\ -\infty & \text{otherwise} \end{cases} \tag{3.1}$$

Notice that $g \in K^{\varnothing} = \mathbb{R}_+^M$ in the definition above. The above definition holds for a technology that satisfies the strong disposability assumption. Notice that the input directional distance function is related to the traditional input distance function $D_i(x, y)$ as follows: $D_L(x, y; g) = 1 - 1/D_i(x, y)$.

Following the traditional duality result result in McFadden (1978) or Shephard (1970) between cost function and input distance function, Luenberger (1992) and Chambers, Chung and Färe (1996) have more recently developed formulations in terms of cost function and input directional distance function. Thus, one can state two duality results making a link between the input directional distance function and the cost function:

$$D_L(x, y; g) = \inf_p \{p.x - C(p, y) : p.g = 1, p \geq 0\}. \tag{3.2}$$

Moreover, we have conversely:

$$C(p, y) = \inf_x \{p.x - p.gD_L(x, y; g) : p \geq 0\}. \tag{3.3}$$

These results hold when $g \neq 0$ and $D_L(x, y; g) > -\infty$. However, these properties only hold over a set $L(y)$ satisfying the strong disposability assumption.

Apart from this traditional duality relationship, a weaker duality result between the cost function and the ray (or weakly) disposable input distance function is available in the literature (e.g., Shephard (1974)) whereby some (but not all) prices are allowed to be negative:[3]

Proposition 3.1.1 *Let L be an input correspondence satisfying L1-L3 and L5. Assume that $g \neq 0$ and $D_L(x, y; g) > -\infty$, we have:*
a) If L4 (convexity) holds then:

$$D_L(x, y; g) = \inf_p \{p.x - C(p, y) : p.g = 1\}. \tag{3.4}$$

b) Let p be an input price vector having some negative components. Assuming that L4 holds, we have:

$$C(p, y) = \inf_x \{p.x - p.gD_L(x, y; g)\}. \tag{3.5}$$

This is an immediate consequence from the fact that a convex set is the intersection of its supporting hyperplanes.

Now, we extend the properties of the directional distance function to account for negative orientations and to be compatible with production sets satisfying the S-disposal assumption.

Proposition 3.1.2 *Let L be an input correspondence satisfying L1-L3. Assume moreover that $L(y)$ satisfies the S-disposal assumption. Assume that $g \neq 0$ and $D_L(x, y; g) > -\infty$, we have:*
a) If L4 (convexity) holds then:

$$D_L(x, y; g) = \inf_p \{p.x - C(p, y) : p.g = 1, p \in \bigcup_{I \in S} K^I\}. \tag{3.6}$$

b) Let $p \in K^I$ be an input price vector having some negative components. Assume that L4 holds, we have:

$$C(p, y) = \inf_x \{p.x - p.gD_L(x, y; g) : x \in L(y)\}. \tag{3.7}$$

Property a) extends the results by Luenberger (1992) and Chambers, Chung and Färe (1996) in the context of an input correspondence that may fail both the strong and the weak disposability assumptions. The converse results expressing the cost function with respect to the directional distance function is stated in b). This duality result considerably weakens current duality results imposing strong disposability (see expressions (3.2) and (3.3)) and weak disposal of inputs (see Proposition 3.1.1), which allow some (but not all) prices to be negative. Otherwise stated, this proposition shows that S-disposal of inputs is a necessary and sufficient condition for the input directional distance

[3] Also McFadden (1978: 60) anticipates the use of negative prices and maintains that duality results can be preserved under these circumstances.

function to characterize technology. This substantially weakens the Färe and Primont (1995) result on the importance of ray disposal in the inputs for the traditional input distance function to characterize technology.

In principle it is possible to relax the convexity assumption. Under non-convexity, the duality result in Proposition 3.1.2 would only hold locally (similar to the local duality result in, e.g., Briec, Kerstens and Vanden Eeckaut (2004)). However, under non-convexity Proposition 3.1.1 would fail to hold, since ray disposal is of little use without convexity.

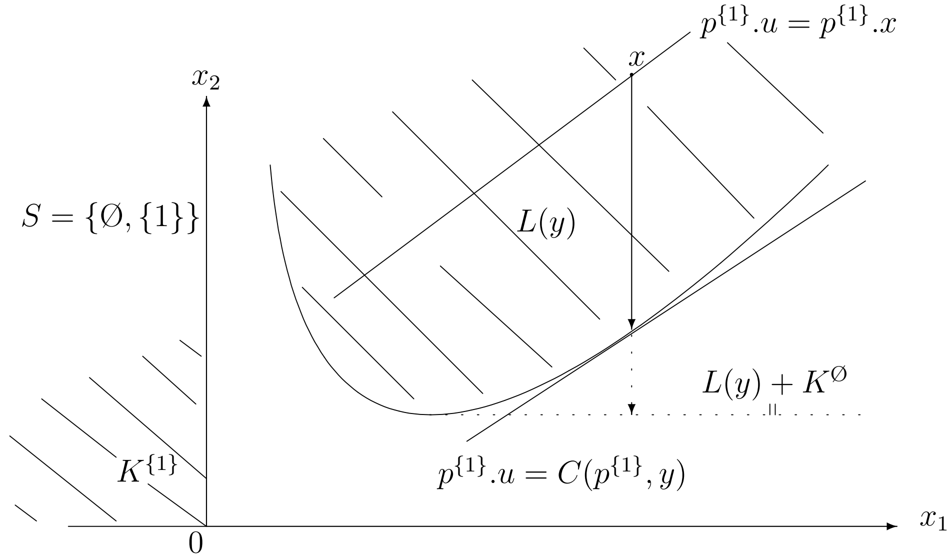

Figure 3.1.1: Directional distance function and duality with $I = \{1\}$ and $S = \{\emptyset, \{1\}\}$.

This new duality result is illustrated in Figure Figure 3.1.1 above for the case $S = \{\emptyset, \{1\}\}$. Since the first input is clearly congested, it receives a negative price and the cost function ends up having a positive rather than a negative slope.

3.2 S-congested and Ray-disposability Congested Technologies: A Comparison

It should be intuitively clear by now that when the input set satisfies free disposability assumptions, then it also satisfies S-disposal assumptions. But, the converse is not true. The same applies to weak disposal assumptions: an input set satisfying weak disposability assumptions also satisfies S-disposal assumptions, but not the converse.

An input set that is weakly disposable can be employed to detect a general form of congestion whereby increasing some inputs decreases outputs (or decreasing inputs increases outputs). An input set satisfying S-disposal assumptions can also detect congestion. This subsection clarifies the link between both approaches to congestion modelling.

Proposition 2.2.5 has a direct implication for the notion of ray disposability as defined by Färe and Grosskopf (1983). When technology is convex and congested in the sense defined by them, then there exists a collection S such that it satisfies a minimal S-disposal assumption and one obtains negative marginal rates of substitution corresponding to the lack of free disposability.

Proposition 3.2.1 *Let L be an input correspondence satisfying L1-L4. If $L(y)$ satisfies L5, but not L6, then we have the following properties:*
a) There exists S that contains $\emptyset$ such that $L(y)$ satisfies a minimal S-disposal assumption.
b) There exists S such that for any $J \in S$, there exists some input price vector $p \in K^J \cap \left(\mathbb{R}^M \setminus \bigcup_{I \in S\setminus\{J\}} K^I\right)$ such that $C(p, y) > -\infty$.

Proposition 3.2.1 characterises ray disposability in the inputs as a special case of minimal S-disposability. Part a) states that any weakly disposable technology can be re-interpreted as an S-disposable technology, but not the converse. Part b) claims that a weakly disposable technology can always be characterised via the support function of its input set. An input set is then ray disposable if there exists a price vector containing some negative prices such that the cost function is bounded.

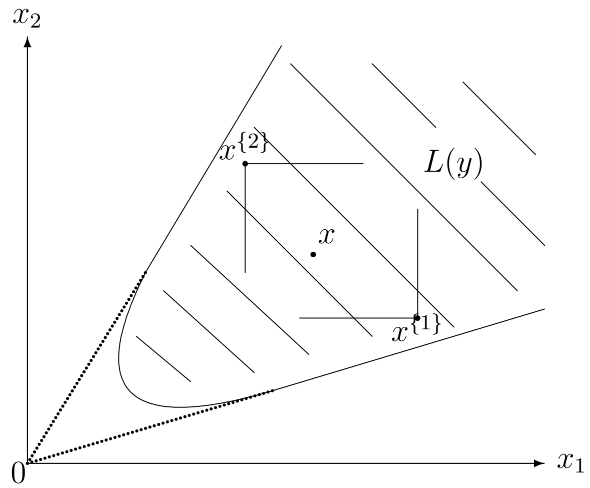

Figure 3.2.1: Weakly Disposable Technologies and S-congestion

The following corollary establishes a link between S-congestion and the notion of congestion defined by Färe and Grosskopf (1983).

Proposition 3.2.2 *Let L be an input correspondence satisfying L1-L4. If $L(y)$ satisfies L5, but not L6, then we have the following properties:*
a) There exists S that contains $\emptyset$ such that $L(y)$ is S-congested.
b) There exists a collection S, $J \notin S$ and some price vector $p \in K^J \cap \left(\mathbb{R}^M \setminus \bigcup_{I \in S\setminus\{J\}} K^I\right)$ such that $C(p, y) > -\infty$.

Thus, for any input set satisfying a certain ray disposal hypothesis, one can always find a corresponding S-congestion assumption that is compatible with the data. Furthermore, the former input set can be reconstructed via a corresponding cost function, just as it is the case for S-congested technologies.

In the following we show that S-congestion can be viewed as a more flexible concept because it can model several forms of congestion as defined in Färe and Svensson (1980). Indeed, in addition to MOL-congestion it also allows to model output prohibitive (OP) congestion and all of its variations in general technologies. Let J be a finite subset of $\{1, \cdots, N\}$. In the following we denote $\mathbb{R}^J_+ = \{y \in \mathbb{R}^N_+ : y_j = 0 \, \forall j \notin J\}$. By definition, we have $\mathbb{R}^J_+ = \mathbb{R}^N_+ \cap K^J$, where K^J is a convex cone of $\mathbb{R}^N$ constructed following equation 2.7.

Definition 3.2.3 *A production technology is said to be OP-congested for the index set J if for all $y \in \mathbb{R}^J_+ \backslash \{0\}$ the input set $L(y)$ is S-bounded.*

It is easy to see that this definition generalizes the OP-congestion as defined for a single-output technologies in Färe and Svensson (1980) to the multiple output case.

The next result establishes that a weak disposable technology cannot satisfy OP axioms.

Proposition 3.2.4 *Let L be an input correspondence satisfying L1-L4. If L satisfies L5, then the technology does not satisfy output prohibitive (OP) congestion.*

By contrast, a S-congested technology may exhibit OP-congestion. This is established in the next example.

Example 3.2.5 *Suppose that $M = N = 1$ and that there exists a continuous function $\phi : \mathbb{R}_+ \Rightarrow \mathbb{R}_+$ such that $T = \{(x, y) \in \mathbb{R}^2_+ : y \leq \phi(x)\}$. Suppose moreover that: a) there exists a unique maximum x^* of ϕ, b) $\lim_{x \to +\infty} \phi(x) = 0$ and c) $\phi(0) = 0$. From a), b and c), this technology satisfies $L1 - L4$ and is clearly OP-congested. Suppose that $x, x' \in L(y)$ and that $u \geq x$ and $u \leq x'$. Since there is a unique maximum and ϕ is continuous, we have $\phi(u) \geq y$. Consequently, $u \in L(y)$ and we conclude that $L(y)$ satisfies a $\{\emptyset, \{1\}\}$-disposal assumption. Moreover, it fails the free disposal assumption ($\emptyset$-disposal assumption) and, consequently, $L(y)$ is $\{\emptyset, \{1\}\}$-congested.*

One could easily extend this example to the multiple output case.

3.3 Directional Distance Function and Congestion Measurement

We are now interested in making the link between the I-oriented directional distance function and the congestion concept. To study this relationship from the dual viewpoint we introduce the adjusted price correspondence $p : \mathbb{R}_+^{M+N} \times \mathbb{R}_+^M \longrightarrow 2^{\mathbb{R}^M}$ due to Luenberger (1995) and defined by:

$$p(x, y; g) = \arg\min \left\{ p.x - C(p, y) : p.g = 1, p \in \bigcup_{I \in S} K^I \right\} \tag{3.8}$$

Notice that if the minimum is not achieved then $p(x, y; g) = \emptyset$. For simplicity, we introduce the following notation:

$$L^{\emptyset}(y) = L(y) + K^{\emptyset} = L(y) + \mathbb{R}_+^M \tag{3.9}$$

$$L^I(y) = L(y) + K^I \tag{3.10}$$

$$L^S(y) = \bigcap_{I \in S} L^I(y) \tag{3.11}$$

Proposition 3.3.1 *Let L be an input correspondence satisfying L1-L3. We have the following properties:*
a) If $L(y)$ is S-congested, then there is $J \notin S$, $g^J \in K^J$ and $x \in W^J(y)$ such that $D_{L^S}(x, y; g^J) = 0$.
b) If $L(y)$ satisfies a minimal S-disposal assumption, then for any $J \in S$, there are $g^J \in K^J$ and $x \in W^J(y)$ such that $D_{L^S}(x, y; g^J) = 0$.
c) If L4 holds, then $L(y)$ is S-congested if and only there exists $J \notin S$ and there are $g^J \in K^J$ and $x \in L(y)$ such that $p(x, y; g^J) \subset K^J$.
d) If L4 holds, then $L(y)$ satisfies a minimal S-disposal assumption if and only if for any $J \in S$ there are $g^J \in K^J$ and $x \in L(y)$ such that $p(x, y; g^J) \in K^J$.
e) If L4 holds, then $L(y)$ is S-congested if and only if there exists $J \notin S$ and some $x \in L(y)$ such that $D_L(x, y; g^J) < D_{L^S}(x, y; g^J)$.
f) If L4 holds, then the S-disposal assumption is minimal if and only if for any $J \in S$ and $x \in L(y)$ $D_L(x, y; g^J) < D_{L^S}(x, y; g^J)$.

Remark that the properties above hold for the general case of a direction vector g possibly having some negative components. However, from a practical standpoint the direction g^I can be chosen such that g^I is non-negative as exemplified in Figure 3.1.1. Proposition 3.3.1 introduces a congestion measure detecting a lack of S-disposability over the input set $L(y)$. Remark that in this case the vector $g^I \in K^I$ is assumed to be non-negative, otherwise our measure would not be operational for measuring congestion.

Definition 3.3.2 *Let L be an input correspondence satisfying L1-L3. Assume that $I \in S$, where S is a collection of subsets of $\{1, ..., M\}$. Let $g^I \in K^I$ be a vector such that*

$$\begin{cases} g_i^I \geq 0 & \text{if } i \notin I \\ g_i^I = 0 & \text{if } i \in I. \end{cases}$$

We define the following congestion measures:
a) The measure defined by $DC^I(x, y, g^I) = D_{L^I}(x, y; g^I) - D_{L(y)}(x, y; g^I)$ *is called* $\{\emptyset, I\}$*-congestion measure.*
b) The measure defined by $ADC_+^S(x, y) = \max_{I \notin S}\{DC(x, y, g^I)\}$ *is called* S*-maximum congestion measure.*
c) The measure defined by $ADC_-^S(x, y) = \min_{I \in S}\{DC(x, y, g^I)\}$ *is called* S*-minimum congestion measure.*

We can now state the following properties for our congestion measures.

Proposition 3.3.3 *Let* L *be an input correspondence satisfying L1-L3. Assume moreover that* $L(y)$ *satisfies the* S*-disposal assumption. Let* $I \subset \{1, ..., M\}$ *such that* $g^I \in K^I$*. We have the following properties:*
a) There exists some $x \in L(y)$ *such that* $DC^I(x, y, g^I) > 0$*, if and only if* $L(y)$ *fails the* $\{\emptyset, I\}$*-disposal assumption.*
b) There exists some $x \in L(y)$ *such that* $ADC_+^S(x, y) > 0$ *if and only if* $L(y)$ *is* S*-congested.*
c) There exists some $x \in L(y)$ *such that* $ADC_-^S(x, y) > 0$ *if and only if* $L(y)$ *satisfies a minimal* S*-disposal assumption.*

The first congestion measure $DC^I(x, y, g^I)$ evaluates eventual congestion componentwise per subset S. The second S-maximum congestion measure takes the maximum of these componentwise congestion measures, thereby revealing the worst congested subset S. Finally, the third S-minimum congestion measure takes the minimum of these componentwise congestion measures, thereby revealing the minimum amount of congestion commonly present among the subsets S.

4 Conclusions

Starting from a relaxed version of the widespread strong disposal assumption we define new technologies capable to model more general notions of congestion. In fact, the S-disposal assumption can be seen as an attempt to re-interpret the traditional strong disposal axiom as a local instead of a global property (an issue already raised in Lau (1974)). These new technologies lead to the formulation of a new duality result between the input directional distance function and the cost function with possibly negative prices. This duality result is considerably weaker than the results available in the current literature. Furthermore, it turns out that the S-disposal assumption allows modeling more general forms of congestion as defined in Färe and Svensson (1980) compared to the ray disposal hypothesis. Indeed, apart from monotone output limitational congestion that can also be represented by ray disposable input sets, technologies with S-disposal of inputs can also model output prohibitive congestion, which cannot be represented by ray disposable input sets.

Straightforward extensions of this contribution are the development of empirical production models capable to test the different disposability assumptions (strong, weak, and S-disposal). This would allow to test whether traditional assumptions like strong and weak disposal of inputs can be maintained against the more general S-disposal assumption. This testing framework could then extend and improve the battery of tests verifying various combinations of strong and weak input disposability in both inputs and outputs as developed by Färe, Grosskopf and Lovell (1987). Furthermore, it could be interesting to see what difference the S-disposal axiom makes compared to the weak disposal hypothesis in terms of the shadow prices for bad outputs when explicitly modeling trade-offs between good and bad outputs in production (e.g., along the lines of Coggins and Swinton (1996)).

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Appendix:

Proof of Proposition 2.2.4: a) Assume that $S' \subset S$. Moreover, suppose that $L(y)$ satisfies the S-disposal assumption. Let a set of input vectors $\left\{x^I\right\}_{I\in S} \subset L(y)$. From Definition 2.2.2, $\forall I \in S$, $x \geq x^I$ implies that $x \in L(y)$. Since $S' \subset S$, $I \in S' \Longrightarrow I \in S$, and thus the S-disposal assumption holds. b) First, assume that $L(y)$ satisfies the S-disposal assumption. Let a set of input vectors $\left\{x^I\right\}_{I\in S} \subset L(y)$. By definition, $\forall I \in S$, $x \geq x^I$ implies that $x \in L(y)$. Consequently, $\forall I \in S$, $x \in L(y) + K^I$. Thus, $x \in \bigcap_{I\in S} L(y) + K^I$ and $L(y) \subset \bigcap_{I\in S} L(y) + K^I$. Conversely, assume that $L(y) = \bigcap_{I\in S} L(y) + K^I$. For any $I \in S$, if $x^I \in L(y)$ and $x \geq^I x^I$, then $x \in L(y) + K^I$. Consequently, let a set of input vectors $\left\{x^I\right\}_{I\in S} \subset L(y)$, $x \in \bigcap_{I\in S} L(y)+K^I = L(y)$. Thus, L(y) satisfies the S-disposal assumption.□

Proof of Proposition 2.2.5: a) We have $L(y) = \bigcap_{I\in S} L(y) + K^I$. But, for any $I \in S$, $L(y) + K^I = \bigcap_{p\in K^I} \left\{x \in \mathbb{R}^M : p.x \geq C(p,y)\right\}$. Consequently,

$$L(y) = \bigcap_{I\in S} \left(\bigcap_{p\in K^I} \left\{x \in \mathbb{R}^M : p.x \geq C(p,y)\right\} \right).$$

This subset can immediately be rewritten as:

$$L(y) = \left\{ x \in \mathbb{R}^M : p.x \geq C(p,y), p \in \bigcup_{I\in S} K^I \right\}.$$

By using Proposition 2.2.4.b, this ends the proof. b) Let $2^{\{1,...,M\}}$ the set of the partitions of $\{1,...,M\}$. Clearly, $\mathbb{R}^M = \bigcup_{I\in 2^{\{1,...,M\}}} K^I$. But, $L(y) = \left\{x \in \mathbb{R}^M : p.x \geq C(p,y), p \in \mathbb{R}^M\right\}$. Thus:

$$L(y) = \left\{ x \in \mathbb{R}^M : p.x \geq C(p,y), p \in \bigcup_{I\in 2^{\{1,...,M\}}} K^I \right\}.$$

Now just take $S = \bigcup_{I\in 2^{\{1,...,M\}}} I$ and there exists a collection S such that $L(y)$ satisfies the S-disposal assumption. Thus, by contradiction and recurrence, it is easy to state that there exists a collection such that the S-disposal assumption is minimal. c) We prove the first implication. If the S-disposal assumption is minimal, we have $L(y) = \bigcap_{I\in S} L(y) + K^I$. We can equivalently write: $L(y) = \bigcap_{I\in S\setminus\{J\}} L(y) + K^I \cap L(y) + K^J$. Assume that $C(p,y) = -\infty$ for any $p \in K^J \cap \left(\mathbb{R}^M \setminus \bigcup_{I\in S\setminus\{J\}} K^I\right)$. This implies that for any $x \in L(y), p.x > C(p,y)$. Consequently, $L(y) = \bigcap_{I \in S \setminus \{J\}} L(y) + K^I$. But, this contradicts that the S-disposability of $L(y)$ is minimal. By contradiction, we deduce that there exists some $p \in K^J \cap \left(\mathbb{R}^M \setminus \bigcup_{I \in S \setminus \{J\}} K^I\right)$ such that $C(p,y) > -\infty$. Now, let us show the converse. Assume that the S-disposability of $L(y)$ is not minimal. Then, there exists S' and J such that $S = S' \cup J$ and such that $L(y)$ satisfies the S'-disposal assumption. Thus, if $L(y)$ satisfies the S-disposal assumption, then $L(y) = \bigcap_{I \in S'} L(y) + K^I$. But since $J \notin S'$, it follows that there exists some $J \in S$ such that if $p \in K^J$ then $C(p,y) = \infty$. Thus, reversing the implication yields the proof. $\square$

Proof of Proposition 2.2.7: a) From Proposition 2.2.4.a, if $S' \subset S$, the S'-disposability assumption implies the S-disposability assumption. Thus, if $L(y)$ fails S-disposability assumption, then it also fails S'-disposability assumption, and this ends the proof. b) If the S-disposability of $L(y)$ is minimal, from Definition 2.2.2, for any $S' \subset S$ with $S \neq S'$, $L(y)$ fails the S'-disposability assumption and thus it is S'-congested. c) Let us prove the first part of the equivalence. Assume that there does not exists some $J \notin S$ and $p^J \in K^J \cap \left(\mathbb{R}^M \setminus \bigcup_{I \in S \setminus \{J\}} K^I\right)$ with $C(p^J, y) > -\infty$. In such a case, since for any $x \in L(y)$ and $I \in S$ given $p \in K^I$ we have $C(p,y) = -\infty$, thus we can write $L(y) = \bigcup_{I \in S} \left\{x : p.x \geq C(p,y), p^I \in K^I\right\}$. Thus, from Proposition 2.2.5, $L(y)$ satisfies the S-disposal assumption and is not S-congested. By contradiction, the first implication is stated. Let us show the converse. If there exists $J \notin S$, and some price vector $p^J \in K^J \cap \left(\mathbb{R}^M \setminus \bigcup_{I \in S \setminus \{J\}} K^I\right)$ such that $C(p^J, y) > -\infty$, it is obvious that $L(y) \neq \bigcap_{I \in S} L(y) + K^I$. Thus, $L(y)$ fails the S-disposability assumption and $L(y)$ is S-congested. $\square$

Proof of Proposition 2.2.9: a) The usual proof to show that the traditional efficient and weak efficient subset are closed can straightforwardly be transposed. b) If the subset $E^J(y)$ is non-empty, clearly $L(y) \neq \bigcap_{J \in S} L(y) + K^J$ and thus $L(y)$ is S-congested. Conversely, if $L(y)$ is S-congested, then there exists some $J \in S$ and some $x \in L(y)$ such that $u \leq^J x$ implies $u \notin L(y)$. Thus, x is J-efficient, and $E^J(y)$ is non-empty. c) is similar by way of Proposition 2.2.5.c. d) and e) are respectively consequences of Proposition 2.2.7.c and Proposition 2.2.5.c.$\square$

Proof of Proposition 2.2.11: a) Let $x \in L(y)$. By hypothesis, for any $I \in S$ there is some $\bar{x}^I$ such that $x_m \geq^I \bar{x}^I_m$. If $\bigcup_{I \in S} I = \{1, ..., M\}$, for any m there is some I such that $x_m \leq x^I_m$. Since this property holds for any $x \in L(y)$, $L(y)$ is bounded. b) If $L(y)$ is S-bounded, then it is easy to show that there exists $I \notin S$, such that $E^I(y)$ is non-empty. Consequently, from Proposition 2.2.9.d, $L(y)$ is S-congested. c) From b), since $L(y)$ is S-bounded for any $I \in S$, $E^I(y)$ is non-empty. Since for any $I \in S$, $E^I(y) \subset W^I(y)$, it follows that $W^I(y)$ is non-empty. But from b) $L(y)$ is S-congested, it follows that from Proposition 2.2.7.c, there exists some $J \in S$ and some price vector $p^J \in K^J \cap \left(\mathbb{R}^M \backslash \bigcup_{I \in S \backslash \{J\}} K^I\right\}$ such that $C(p^J, y) > -\infty$. Clearly, since $p^J \in K^J \cap \left(\mathbb{R}^M \backslash \bigcup_{I \in S \backslash \{J\}} K^I\right\}$, the cost function is achieved by some $x \in W^J(y)$, and this terminates the proof. □

Proof of Proposition 3.1.1: From Luenberger (1992) and Chambers, Chung and Färe (1996) the proof of a) and b) is immediate using the fact that $L(y)$ is convex which implies that:

$$\begin{aligned} L(y) &= \left\{x \in \mathbb{R}^M : p.x \geq C(p, y), p \in \mathbb{R}^M\right\} \\ &= \bigcap_{p \in \mathbb{R}^M} \{x \in \mathbb{R}^M : p.x \geq C(p, y)\}. \quad \square \end{aligned}$$

Proof of Proposition 3.1.2: a) We have $D_L(x, y; g) = \inf\{\delta : x - \delta g \in \mathbb{R}^M \backslash L(y)\}$. But from Proposition 2.2.5.a:

$$\begin{aligned} L(y) &= \left\{x \in \mathbb{R}^M : p.x \geq C(p, y), p \in \bigcup_{I \in S} K^I\right\} \\ &= \bigcap_{p \in \bigcup_{I \in S} K^I} \{x \in \mathbb{R}^M : p.x \geq C(p, y)\}. \end{aligned}$$

Thus:

$$\begin{aligned} \mathbb{R}^M \backslash L(y) &= \bigcup_{p \in \bigcup_{I \in S} K^I} \mathbb{R}^M \backslash \{x \in \mathbb{R}^M : p.x \geq C(p, y)\} \\ &= \bigcup_{p \in \bigcup_{I \in S} K^I} \{x \in \mathbb{R}^M : p.x < C(p, y)\}. \end{aligned}$$

Let us denote:

$$\delta_p = \inf\{\delta : p.(x - \delta g) < C(p, y)\}.$$

Now, we have the equality $D_L(x, y; g) = \inf\{\delta : x - \delta g \in \mathbb{R}^M \backslash L(y)\} = \inf_{p \in \bigcup_{I \in S} K^I} \inf_\delta \left\{\delta : x - \delta g \in \{u \in \mathbb{R}^M : p.u < C(p, y)\}\right\} = \inf_{p \in \bigcup_{I \in S} K^I} \delta_p$. If $p.g \neq 0$, then an elementary calculus yields:

$$\delta_p = \frac{p.x - C(p, y)}{p.g}.$$

Moreover, if $p.g = 0$, then:

$$\delta_p = \begin{cases} +\infty & \text{if} \quad p.x \geq C(p, y) \\ -\infty & \text{if} \quad p.x < C(p, y). \end{cases}$$

However, since by hypothesis $D_L(x, y; g) > -\infty$, there is some $\delta \in \mathbb{R}$ such that $x - \delta g \in L(y)$. Thus, for all price vectors p, we have $\delta_p > -\infty$ and the second case is excluded. Consequently, we deduce that:

$$\inf_{p \in \bigcup_{I \in S} K^I} \delta_p = \inf_{p \in \bigcup_{I \in S} K^I} \left\{\frac{p.x - C(p, y)}{p.g}\right\}.$$

Hence, we deduce that: $D_L(x, y; g) = \inf_{p \in \bigcup_{I \in S} K^I} \left\{ \frac{p.x - C(p,y)}{p.g} \right\}$ which yields the result, under a suitable normalization. b) can be obtained in a way similar to Luenberger (1992). □

Proof of Proposition 3.2.1: a) Since $L(y)$ is convex, from Proposition 2.2.5.b, the result is immediate. b) follows from a). □

Proof of Proposition 3.2.2: The proof is derived from Proposition 2.2.7. a) Since $L(y)$ is convex, from Proposition 2.2.7.b, the result is immediate. b) follows from a). □

Proof of Proposition 3.2.4: Let us consider $y \neq 0$ such that $L(y) \neq \emptyset$. In such a case there exists $x \in L(y)$ such that $x \neq 0$. From the weak disposal assumption, we have $\{tx : t \geq 1\} \subset L(y)$. However, the subset $\{tx : t \geq 1\}$ is not bounded. Consequently, $L(y)$ is not bounded. Therefore, the technology is not OP-congested for all $J \subset \{1, ..., N\}$. □

Proof of Proposition 3.3.1 : a) is a straightforward consequence of Proposition 2.2.5.c. Similarly, b) follows from Proposition 2.2.5.c. c) follows from a) and from the duality result in Proposition 3.1.2.b. Similarly, d) follows from b) and from the duality result in Proposition 3.1.2.b. e) If $L(y)$ is S-congested, it fails the S-disposal assumption. Consequently, there exists $J \notin S$ such that $L^J(y) = L(y) + K^J \neq L(y)$. Thus, there exists $\hat{x} \in L^J(y) \backslash L(y)$. Let $\hat{\delta} = \arg\max\{\delta : \hat{x} - \delta g^J \in L(y)\}$. Obviously, $\hat{\delta} < 0$. Thus, $D_{L^J}\left(\hat{x} - \hat{\delta} g^J, y; g^J\right) > 0$. Since $D_L\left(\hat{x} - \hat{\delta} g^J, y; g^J\right) = 0$, e) is obtained. f) The property is obtained in a similar way. □

Proof of Proposition 3.3.3: a) As a straightforward consequence $L(y) \neq L(y) + K^I$. By definition, the result is immediate. b) follows from Proposition 3.3.1.d. c) follows immediately from Proposition 3.3.1.e. □